AD-A096 356 NAVAL SURFACE WEAPONS CENTER DAHLGREN VA F/G 8/5
DECREASING THE INFLUENCE OF DISTANT ZONES WITH MODIFICATIONS TO--ETC(U)
JAN 81 P FELL, M KARASKA
UNCLASSIFIED NSWC/TR-80-350 NL

1 OF 1
AD A
096356

END
DATE
FILMED
4-81
DTIC

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NSWC/TR-80-350	2. GOVT ACCESSION NO. AD-A096 356	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) DECREASING THE INFLUENCE OF DISTANT ZONES WITH MODIFICATIONS TO THE STOKES AND VENING MEINESZ KERNELS		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Patrick Fell Maksiminas Karaska		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Surface Weapons Center (K10) Dahlgren, Virginia 22448		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Strategic Systems Project Office (SP-23) Washington, D. C. 20376		12. REPORT DATE Janu[redacted]81 13. NUMBER OF PAGES 58
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

geoid undulation
vertical deflection
truncation error
Stokes integral
Vening Meinesz integral

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The expected influence of distant zones on the computations of geoid undulation and vertical deflection is dramatically reduced with the application of a minimization procedure proposed by Molodenskii. Assuming that the lower degree harmonics of the gravity anomaly field are known, results are obtained for the Stokes and Vening Meinesz integrals. The results indicate that the expected truncation error based on these optimized kernel functions is negligible for typical applications when relatively few harmonics are known.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

FOREWORD

This work was performed in the Space and Ocean Geodesy Branch, Space and Surface Systems Division of the Naval Surface Weapons Center under sponsorship of the Navy Strategic Systems Project Office (SP-23).

Released by:

R. T. Ryland, Head
Strategic Systems Department

A

CONTENTS

PRECEDING PAGE BLANK-NOT FILMED

1. INTRODUCTION

If a gravimetric quantity X is a function of $\zeta(\psi, \alpha)$ of the form

$$X(P) = \beta \iint_{\sigma} K(\psi)\, \zeta(\psi, \alpha)\, d\sigma \tag{1.1}$$

where $\zeta(\psi, \alpha)$ is some measured quantity related to the gravitational potential, then failure to integrate equation (1.1) over the entire sphere introduces an error δX into the computation. If the region of integration is confined to a spherical cap centered on the computation point P, the error is usually denoted as the cap truncation error. Molodenskii et al. (1962) and recently Dickson (1979) have given a method for minimizing δN for the Stokes integral. The method assumes that the first M harmonics of $\Delta g(\psi, \alpha)$ are known a priori so that one may replace $S(\psi)$ with

$$S_M(\psi) = S(\psi) - L_M(\psi) \tag{1.2}$$

where

$$L_M(\psi) = \sum_{k=0}^{M} a_k^0\, P_k(\cos\psi). \tag{1.3}$$

The functions $P_k(\cos\psi)$ are the Legendre polynomials and the coefficients a_k are determined by minimizing the integral

$$\int_{\psi_0}^{\pi} [S(\psi) - L_M(\psi)]^2 \sin\psi\, d\psi \tag{1.4}$$

In this report this method will be used to reduce the cap truncation error for the Stokes equation, then modified and applied to the Vening Meinesz integral.

2. APPLICATION OF THE MOLODENSKII PROCEDURE TO THE STOKES INTEGRAL

Consider first the Stokes equation

$$N(P) = \frac{R}{4\pi G} \int_0^{2\pi} \int_0^{\pi} S(\psi)\, \Delta g(\psi, \alpha) \sin \psi \, d\psi \, d\alpha \tag{2.1}$$

having the kernel function

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n + 1}{n - 1} P_n(\cos \psi). \tag{2.2}$$

If the integration of equation (2.1) is extended only up to a spherical distance ψ_0 from the point P, then equation (2.1) may be written as

$$N(P) = \frac{R}{4\pi G} \int_0^{2\pi} \int_0^{\psi_0} S(\psi)\, \Delta g(\psi, \alpha)\, do + \delta N_l \tag{2.3}$$

where do is the incremental surface element and δN_l is the cap truncation error defined as

$$\delta N_l = \frac{R}{4\pi G} \int_0^{2\pi} \int_{\psi_0}^{\pi} S(\psi)\, \Delta g(\psi, \alpha)\, dc. \tag{2.4}$$

The subscript l follows the usage in [Fell, 1978]. Following the practice [Heiskanen & Moritz, 1967] of defining the truncation function as

$$\overline{S}(\psi) = \begin{cases} 0 & \text{if } 0 \leqslant \psi < \psi_0 \\ S(\psi) & \text{if } \psi_0 \leqslant \psi \leqslant \pi \end{cases} \tag{2.5}$$

and expanding it into a series of Legendre polynomials

$$\overline{S}(\psi) = \sum_{n=0}^{\infty} \frac{2n + 1}{2} Q_n \; P_n(\cos \psi) \tag{2.6}$$

allows the truncation error to be expressed as

$$\delta N_l = \frac{R}{4\pi G} \sum_{n=0}^{\infty} \frac{2n + 1}{2} Q_n \int_0^{2\pi} \int_0^{\pi} \Delta g(\psi, \alpha)\, P_n(\cos \psi)\, do. \tag{2.7}$$

The double integral is equal to $4\pi \Delta g_n/(2n + 1)$, so that

$$\delta N_l = \frac{R}{2G} \sum_{n=2}^{\infty} Q_n \Delta g_n \tag{2.7}$$

where Δg_n is the n'th degree harmonic from the expansion

$$\Delta g = \sum_{n=2}^{\infty} \Delta g_n(\theta, \lambda) \tag{2.8}$$

and the Q_n are the Molodenskii coefficients defined as

$$Q_n = Q_n(\psi_0) \equiv \int_0^\pi \bar{S}(\psi)\, P_n(\cos\psi) \sin\psi d\psi$$

$$= \int_{\psi_0}^\pi S(\psi)\, P_n(\cos\psi) \sin\psi d\psi \tag{2.9}$$

The rms error due to neglecting distant zones is the square root of

$$\overline{\delta N_t^2} = \frac{R^2}{4G^2} \sum_{n=2}^{\infty} Q_n^2 c_n \tag{2.10}$$

where the c_n's are the anomaly degree variances [Heiskanen and Moritz, 1967].

Under the assumption that the first M harmonics of Δg are known, define

$$\widetilde{\Delta g} = \Delta g - \sum_{n=2}^{M} \Delta g_n = \sum_{n=M+1}^{\infty} \Delta g_n \tag{2.11}$$

Then equation (2.1) may be written as

$$N = \frac{R}{4\pi G} \sum_{n=2}^{M} \iint_{\sigma} \Delta g_n\, S(\psi)\, d\sigma + \frac{R}{4\pi G} \iint_{\sigma} \widetilde{\Delta g}\, S(\psi)\, d\sigma \tag{2.12}$$

where the integration is carried out over the sphere.

The first integral in equation (2.12) reduces to

$$\frac{R}{G} \sum_{n=2}^{M} \frac{\Delta g_n}{n-1} = N_2 + N_3 + \ldots + N_M \tag{2.13}$$

where $\Delta g_2, \ldots, \Delta g_M$ are assumed given.

Let

$$S_M(\psi) = S(\psi) - L_M(\psi) \tag{2.14}$$

where

$$L_M(\psi) = \sum_{k=0}^{M} a_k^0 P_k(\cos\psi) \tag{2.15}$$

and replace $S(\psi)$ by $S_M(\psi)$ in the second integral of equation (2.12).

From the orthogonality properly of Legendre polynomials, the second integral is zero for $k \leqslant M$. Thus $S(\psi)$ can be suitably modified in this fashion without affecting the value of N. Thus

$$N = N_2 + \ldots + N_M + \frac{R}{4\pi G} \iint_{\sigma} \widetilde{\Delta g}\, S_M(\psi)\, d\sigma$$

$$= N_2 + \ldots + N_M + \frac{R}{4\pi G} \int_0^{2\pi} \int_0^{\psi_0} \widetilde{\Delta g}\, S_M(\psi)\, d\sigma + \delta N_M \tag{2.16}$$

where δN_M is the cap truncation error under the assumptions made above,

$$\delta N_M = \frac{R}{4\pi G} \int_0^{2\pi} \int_{\psi_0}^{\pi} \widetilde{\Delta g}\, S_M(\psi)\, d\sigma. \tag{2.17}$$

Let

$$\overline{S}_M(\psi) = \begin{cases} 0 & \text{if } 0 \leqslant \psi < \psi_0 \\ S_M(\psi) & \text{if } \psi_0 \leqslant \psi \leqslant \pi \end{cases} \tag{2.18}$$

and expand it into a series of Legendre polynomials

$$\overline{S}_M(\psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} q_n^0 P_n(\cos\psi) \tag{2.19}$$

where

$$q_n^0 = q_n^0(\psi_0) = \int_0^\pi \overline{S}_M(\psi)\, P_n(\cos\psi) \sin\psi d\psi$$

$$= \int_{\psi_0}^{\pi} S_M(\psi)\, P_n(\cos\psi) \sin\psi d\psi. \tag{2.20}$$

Using equations (2.14) and (2.15) and letting

$$R_{nk}^0 = \int_{\psi_0}^{\pi} P_n(\cos\psi)\, P_k(\cos\psi) \sin\psi d\psi \tag{2.21}$$

one can rewrite equation (2.20) as

$$q_n^0 = Q_n - \sum_{k=0}^{M} a_k^0 R_{nk}^0 \tag{2.22}$$

and the cap truncation error as

$$\delta N_M = \frac{R}{4\pi G} \sum_{n=0}^{\infty} \frac{2n+1}{2} q_n^0 \int_0^{2\pi}\int_0^{\pi} \widetilde{\Delta g}\, P_n(\cos\psi) \sin\psi d\psi d\alpha. \tag{2.23}$$

Since the integral in equation (2.23) is zero for $n \leqslant M$ and equal to $4\pi\Delta g_n/(2n+1)$ for $n \geqslant M+1$,

$$\delta N_M = \frac{R}{2G} \sum_{n=M+1}^{\infty} q_n^0 \Delta g_n. \tag{2.24}$$

The rms error due to neglecting distant zones is the square root of

$$\overline{\delta N_M^2} = \frac{R^2}{4G^2} \sum_{n=M+1}^{\infty} (q_n^0)^2 c_n \tag{2.25}$$

To evaluate equation (2.25) the coefficients a_k^0 must be computed. This is done by applying Schwarz's inequality to equation (2.17),

$$\left[\iint_\sigma \widetilde{\Delta g}\, S_M(\psi)\, d\sigma\right]^2 \leqslant \iint_\sigma [\widetilde{\Delta g}]^2\, d\sigma \cdot \iint_\sigma [S_M(\psi)]^2\, d\sigma. \tag{2.26}$$

and minimizing the second integral on the right. This leads to the following set of $M + 1$ equations with as many unknowns

$$\frac{\partial}{\partial a_n^0}\left\{\int_0^{2\pi}\int_{\psi_0}^{\pi}\left[S(\psi) - \sum_{k=0}^{M} a_k^0\, P_k(\cos\psi)\right]^2 do\right\} = 0 \qquad \text{for } n = 0, 1, \ldots, M \tag{2.27}$$

giving

$$\int_{\psi_0}^{\pi} S(\psi)\, P_n(\psi)\sin\psi d\psi = \sum_{k=0}^{M} a_k^0 \int_{\psi_0}^{\pi} P_n(\cos\psi)\, P_k(\cos\psi)\sin\psi d\psi.$$

Using equation (2.9)

$$Q_n = \sum_{k=0}^{M} a_k^0 R_{nk}^0 \qquad \text{for } n = 0, 1, \ldots, M. \tag{2.28}$$

Expressing equation (2.28) in matrix notation

$$\{Q\} = [R^0]\,\{a^0\}$$

gives the solution as

$$\{a^0\} = [R^0]^{-1}\,\{Q\}.$$

Note that $a_k^0 = a_k^0(M, \psi_0)$, i.e., a_k^0 is a function of M, the number of lower degree harmonics assumed known, and of the cap size, ψ_0. Furthermore, $q_n^0 = 0$ for $n \leqslant M$.

3. APPLICATION OF THE MOLODENSKII PROCEDURE TO THE VENING MEINESZ INTEGRAL

The two components of the deflection of the vertical are the north-south component ξ and the east-west component η. The Vening Meinesz integral

$$\left\{ \begin{matrix} \xi \\ \eta \end{matrix} \right\} = \frac{1}{4\pi G} \int_0^{2\pi} \int_0^{\pi} S'(\psi)\, \Delta g(\psi, \alpha) \left\{ \begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix} \right\} d\sigma \tag{3.1}$$

where

$$S'(\psi) = \frac{d}{d\psi} S(\psi) = -\sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n^1(\cos\psi) \tag{3.2}$$

permits the computation of the deflection of the vertical from gravity anomaly. If the integration is carried out over a spherical cap of radius ψ_0, then

$$\left\{ \begin{matrix} \xi \\ \eta \end{matrix} \right\} = \frac{1}{4\pi G} \int_0^{2\pi} \int_0^{\psi_0} S'(\psi)\, \Delta g \left\{ \begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix} \right\} d\sigma + \left\{ \begin{matrix} \delta\xi \\ \delta\eta \end{matrix} \right\}_t \tag{3.3}$$

where

$$\left\{ \begin{matrix} \delta\xi \\ \delta\eta \end{matrix} \right\}_t = \frac{1}{4\pi G} \int_0^{2\pi} \int_{\psi_0}^{\pi} S'(\psi)\, \Delta g \left\{ \begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix} \right\} d\sigma \tag{3.4}$$

is the cap truncation error.

Following [Hagiwara, 1973] introduce the function

$$\bar{\zeta}(\psi) = \begin{cases} 0 & \text{if } 0 \leqslant \psi < \psi_0 \\ \frac{1}{2} S'(\psi) & \text{if } \psi_0 \leqslant \psi \leqslant \pi \end{cases} \tag{3.5}$$

and expand it into a series of associated Legendre functions

$$\bar{\zeta}(\psi) = \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} q_n(\psi_0) P_n^1(\cos\psi) \tag{3.6}$$

where $q_n(\psi_0)$ are the Cooke coefficients defined by

$$q_n(\psi_0) = \frac{1}{2} \int_{\psi_0}^{\pi} S'(\psi) P_n^1(\cos\psi) \sin\psi d\psi. \tag{3.7}$$

Furthermore, Hagiwara shows that $q_n(\psi_0)$ and $Q_n(\psi_0)$ are related by

$$q_n(\psi_0) = -\frac{n(n+1)}{2} Q_n(\psi_0) - \frac{1}{2} S(\psi_0) P_n^1(\cos\psi_0) \sin\psi_0. \tag{3.8}$$

Using equations (3.5) and (3.6) and defining

$$Q_n^* = Q_n^*(\psi_0) = -\frac{2}{n(n+1)} q_n(\psi_0)$$

$$= Q_n + \frac{1}{n(n+1)} S(\psi_0) P_n^1(\cos\psi_0)\sin\psi_0 \tag{3.9}$$

the cap truncation error, equation (3.4), becomes

$$\left\{ \begin{matrix} \delta\xi \\ \delta\eta \end{matrix} \right\}_t = -\frac{1}{2\pi G} \sum_{n=1}^{\infty} \frac{2n+1}{4} Q_n^* \int_0^{2\pi}\int_0^{\pi} \Delta g\, P_n^1(\cos\psi) \left\{ \begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix} \right\} do. \tag{3.10}$$

It can be shown [Hagiwara, 1973], [deWitte, 1967] that the double integral in equation (3.10) through a change to latitude and longitude variables is equal to

$$\int_0^{2\pi}\int_0^{\pi} \Delta g\, P_n^1(\cos\psi) \left\{ \begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix} \right\} do = \frac{4\pi}{2n+1} \left\{ \begin{matrix} \dfrac{\partial}{\partial\phi}\Delta g_n \\ \dfrac{1}{\cos\phi}\dfrac{\partial}{\partial\lambda}\Delta g_n \end{matrix} \right\}. \tag{3.11}$$

So the cap truncation error becomes

$$\left\{ \begin{matrix} \delta\xi \\ \delta\eta \end{matrix} \right\}_t = -\frac{1}{2G} \sum_{n=2}^{\infty} Q_n^* \left\{ \begin{matrix} \dfrac{\partial}{\partial\phi}\Delta g_n \\ \dfrac{1}{\cos\phi}\dfrac{\partial}{\partial\lambda}\Delta g_n \end{matrix} \right\}. \tag{3.12}$$

and the rms error due to neglecting distant zones is the square root of

$$\overline{\delta\theta_t^2} = E\{\delta\xi^2 + \delta\eta^2\} = \frac{1}{4G^2} \sum_{n=2}^{\infty} n(n+1) Q_n^{*2} c_n \tag{3.13}$$

where E is the expected value operator.

Following a procedure similiar to that used for the Stokes integral, assume that the first M harmonics of Δg are known.

Let

$$\zeta_M(\psi) = S'(\psi) - L_M^1(\psi) \tag{3.14}$$

where

$$L_M^1(\psi) = \sum_{k=1}^{M} a_k^1 P_k^1(\cos\psi), \tag{3.15}$$

and in equation (3.1) replace $S'(\psi)$ by $\zeta_M(\psi)$. This yields

$$\left\{ \begin{matrix} \xi \\ \eta \end{matrix} \right\} = \frac{1}{4\pi G} \sum_{n=2}^{M} \int_0^{2\pi}\int_0^{\pi} \Delta g_n S'(\psi) \left\{ \begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix} \right\} do \tag{3.16}$$

$$+ \frac{1}{4\pi G} \int_0^{2\pi}\int_0^{\psi_0} \widetilde{\Delta g}\, \zeta_M(\psi) \left\{ \begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix} \right\} do + \left\{ \begin{matrix} \delta\xi \\ \delta\eta \end{matrix} \right\}_M .$$

Using equation (3.11) the first integral in equation (3.16) can be evaluated giving

$$-\frac{1}{G}\sum_{n=2}^{M}\frac{1}{(n-1)}\left\{\begin{array}{c}\frac{\partial}{\partial\phi}\Delta g_n\\ \frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\Delta g_n\end{array}\right\}. \tag{3.17}$$

The last term in equation (3.16) is the cap truncation error

$$\left\{\begin{array}{c}d\xi\\ d\eta\end{array}\right\}_M = \frac{1}{4\pi G}\int_0^{2\pi}\int_{\psi_0}^{\pi}\widetilde{\Delta g}\,\zeta_M(\psi)\left\{\begin{array}{c}\cos\alpha\\ \sin\alpha\end{array}\right\}d\sigma. \tag{3.18}$$

Let

$$\bar{\zeta}_M(\psi) = \begin{cases} 0 & \text{if } 0 \leqslant \psi < \psi_0 \\ \frac{1}{2}\,\zeta_M(\psi) & \text{if } \psi_0 \leqslant \psi \leqslant \pi \end{cases} \tag{3.19}$$

and expand it in a series of associated Legendre functions of order one,

$$\bar{\zeta}_M(\psi) = \sum_{n=1}^{\infty}\frac{2n+1}{2n(n+1)}\,\tilde{q}_n\,P_n^1(\cos\psi) \tag{3.20}$$

where

$$\tilde{q}_n \equiv \tilde{q}_n(\psi_0) \equiv \int_0^{\pi}\bar{\zeta}_M(\psi)\,P_n^1(\cos\psi)\sin\psi d\psi$$

$$= \frac{1}{2}\int_{\psi_0}^{\pi}\zeta_M(\psi)\,P_n^1(\cos\psi)\sin\psi d\psi. \tag{3.21}$$

Using equations (3.9) and (3.15) and defining

$$R_{nk}^1 \equiv \int_{\psi_0}^{\pi}P_n^1(\cos\psi)\,P_k^1(\cos\psi)\sin\psi d\psi \tag{3.22}$$

$$q_n^1 \equiv q_n^1(\psi_0) \equiv Q_n^* + \frac{1}{n(n+1)}\sum_{n=1}^{M}a_k^1 R_{nk}^1 \tag{3.23}$$

the expressions for $\tilde{q}_n(\psi_0)$ and $\bar{\zeta}_M(\psi)$ become

$$\tilde{q}_n = q_n - \frac{1}{2}\sum_{k=1}^{M}a_k^1 R_{nk}^1$$

$$= -\frac{n(n+1)}{2}\left[Q_n^* + \frac{1}{n(n+1)}\sum_{k=1}^{M}a_k^1 R_{nk}^1\right]$$

$$= -\frac{n(n+1)}{2}\,q_n^1 \tag{3.24}$$

and

$$\tilde{\zeta}_M(\psi) = -\sum_{n=1}^{\infty} \frac{2n+1}{4} q^1_n P_n^1(\cos\psi). \tag{3.25}$$

From equation (3.9) we note that the q_n^1 are related to the Molodenskii coefficients

$$q_n^1 = Q_n + \frac{1}{n(n+1)} S(\psi_0)\, P_n^1(\cos\psi_0) \sin\psi_0 + \frac{1}{n(n+1)} \sum_{k=1}^{M} a_k^1 R^1_{nk} \tag{3.26}$$

The cap truncation error, equation (3.18) can now be expressed as

$$\begin{aligned} \left\{ \begin{matrix} d\xi \\ d\eta \end{matrix} \right\}_M &= -\frac{1}{2\pi G} \sum_{n=1}^{\infty} \frac{(2n+1)}{4} q_n^1 \int_0^{2\pi}\!\!\int_0^{\pi} \widetilde{\Delta g}\, P_n^1(\cos\psi) \left\{ \begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix} \right\} d\sigma \\ &= -\frac{1}{2G} \sum_{n=M+1}^{\infty} q_n^1 \left\{ \begin{matrix} \dfrac{\partial}{\partial\phi} \Delta g_n \\ \dfrac{1}{\cos\phi} \dfrac{\partial}{\partial\lambda} \Delta g_n \end{matrix} \right\} \end{aligned} \tag{3.27}$$

and the total rms error due to neglecting distant zones as the square root of

$$\overline{d\theta}^2_M = \frac{1}{4G^2} \sum_{n=M+1}^{\infty} n(n+1)[q_n^1]^2 c_n. \tag{3.28}$$

Before computing the coefficients q_n^1, we need to determine the coefficients a_k^1. Applying to equation (3.18) the same procedure as was used for Stokes integral leads to the condition

$$\frac{\partial}{\partial a_n^1} \left\{ \int_0^{2\pi}\!\!\int_{\psi_0}^{\pi} \left[S'(\psi) - \sum_{k=1}^{M} a_k^1 P_k^1(\cos\psi) \right]^2 d\sigma \right\} = 0, \text{ for } n = 1, 2, \ldots, M \tag{3.29}$$

which results in the linear equations

$$\int_{\psi_0}^{\pi} S'(\psi)\, P_n^1(\cos\psi) \sin\psi d\psi = \sum_{k=1}^{M} a_k^1 \int_{\psi_0}^{\pi} P_n^1(\cos\psi)\, P_k^1(\cos\psi) \sin\psi d\psi, \text{ for } n = 1, 2, \ldots, M \tag{3.30}$$

or

$$q_n = \frac{1}{2} \sum_{k=1}^{M} a_k^1 R^1_{nk}, \quad \text{for } n = 1, 2, \ldots, M. \tag{3.31}$$

Expressing equation (3.31) in matrix notation gives

$$\{q\} = \frac{1}{2}\,[R^1]\,\{a^1\}$$

giving the solution

$$\{a^1\} = 2[R^1]^{-1}\{q\}. \tag{3.32}$$

Note that $a_k^1 = a_k^1(M, \psi_0)$ and from equation (3.24)

$$\tilde{q}_n = q_n^1 = 0 \text{ for } n \leqslant M.$$

4. COMPUTATIONAL APPROACH FOR EVALUATING THE TRUNCATION ERROR, RESULTS, AND CONCLUSIONS

The truncation error for the Stokes and Vening Meinesz integrals due to neglecting distant zones was given by equations (2.10) and (3.13) respectively. Introducing the modification to the kernel recommended by Molodenskii (1962) leads to equation (2.25) for geoid height. For vertical deflection an analogous procedure leads to equation (3.28).

Other methods for minimizing the truncation error have been studied [Wong (1969), Meissl (1971), Fell (1978)]. In addition to the Molodenskii procedure, results in this report include the procedure recommended by Fell (1978) for the Stokes kernel, consisting of harmonic removal from the gravity anomaly field alone. Results from this procedure are presented for both the Stokes and Vening Meinesz integrals.

For this latter method the expected truncation error for geoid height is given by the equation

$$\overline{\delta N^2_{III}} = \frac{R^2}{4G^2} \sum_{n=M+1}^{\infty} Q_n^2 c_n \tag{4.1}$$

where the Q_n are the Molodenskii coefficients given by equation (2.9). For vertical deflection the expected truncation error for this method is

$$\overline{\delta\theta^2_{III}} = \frac{1}{4G^2} \sum_{n=M+1}^{\infty} n(n+1) Q_n^{*2} c_n \tag{4.2}$$

where the Q_n^* coefficients are given by equation (3.9) as developed by Hagiwara (1973).

In addition to equations (4.1) and (4.2) the equations used to evaluate the truncation errors are summarized as follows:

$$\overline{\delta N^2_I} = \frac{R^2}{4\pi G^2} \sum_{n=2}^{\infty} Q_n^2 c_n \tag{4.3}$$

$$\overline{\delta N^2_M} = \frac{R^2}{4\pi G^2} \sum_{n=M+1}^{\infty} (q_n^0)^2 c_n \tag{4.4}$$

$$\overline{\delta\theta^2_I} = \frac{1}{4G^2} \sum_{n=2}^{\infty} n(n+1) Q_n^{*2} c_n \tag{4.5}$$

$$\overline{\delta\theta^2_M} = \frac{1}{4G^2} \sum_{n=M+1}^{\infty} n(n+1)(q_n^1)^2 c_n \tag{4.6}$$

where

$$Q_n = \int_{-1}^{x_0} S(x) P_n(x)\, dx \tag{4.7}$$

$$q_n^0 = Q_n - \sum_{k=0}^{M} a_k^0 R_{nk}^0 \tag{4.8}$$

$$R_{nk}^0 = \int_{-1}^{x_0} P_n(x) P_k(x)\, dx \tag{4.9}$$

$$Q_n^* = Q_n + \frac{1}{(n+1)} S(x_0)[P_{n-1}(x_0) - x_0 P_n(x_0)] \tag{4.10}$$

$$q_n^1 = Q_n^* + \frac{1}{n(n+1)} \sum_{k=1}^{M} a_k^1 R_{nk}^1 \tag{4.11}$$

$$R^1_{nk} = \int_{-1}^{x_0} P^1_n(x)\, P^1_k(x)\, dx \tag{4.12}$$

$$S(x_0) = \frac{1}{\sin \frac{\psi_0}{2}} - 6 \sin \frac{\psi_0}{2} + 1 - \cos \psi_0 \;[5 - 3 \ln(\sin \frac{\psi_0}{2} + \sin^2 \frac{\psi_0}{2})] \tag{4.13}$$

$$x_0 = \cos \psi_0 \tag{4.14}$$

All computations were done on the CDC-6700 computer using double precision arithmetic. The anomaly degree variances were computed using Kaula's rule

$$c_n = \frac{192}{n + 1.5} \text{ for } n \geqslant 3 \tag{4.15}$$

with c_2 taken as 10 mgals^2.

The values of several constants were taken as

$$G = 9.798 \times 10^5 \text{ mgals}$$

$$R^2/G^2 = 42.3\ \text{m}^2/\text{mgal}^2. \tag{4.16}$$

The Molodenskii coefficients, Q_n, were computed using the recurrence relations derived by Paul (1973). To compute certain other quantities, the following identities were used (See Appendix C for derivation).

$$\int_a^b [P_n(x)]^2\, dx = \frac{2n-1}{2n+1} \int_a^b P^2_{n-1}(x)\, dx + \left[\frac{xP^2_n(x) + P^2_{n-1}(x) - 2P_n(x)\, P_{n-1}(x)}{(2n+1)} \right]_a^b \text{ for } n \geqslant 1 \tag{4.17}$$

$$\int_a^b P_n(x)\, P_k(x)\, dx = \left[\frac{(n-k)\, x\, P_n(x)\, P_k(x) - nP_{n-1}(x)\, P_k(x) + kP_n(x)\, P_{k-1}(x)}{(n-k)(n+k+1)} \right]_a^b$$

$$\text{for } n \neq k \text{ and } n \geqslant 1,\ k \geqslant 1 \tag{4.18}$$

$$\int_a^b [P^1_n(x)]^2\, dx = n(n+1) \int_a^b [P_n(x)]^2\, dx + n[P_n(x)\, P_{n-1}(x) - xP^2_n(x)]_a^b \qquad \text{for } n \geqslant 1 \tag{4.19}$$

$$\int_a^b P^1_k(x)\, P^1_n(x)\, dx = -nk \left[\frac{(n-k)xP_n(x)\, P_k(x) - (n+1)\, P_n(x)\, P_{k-1}(x) + (k+1)\, P_{n-1}(x)\, P_k(x)}{(n-k)(n+k+1)} \right]_a^b$$

$$\text{for } n \neq k \text{ and } n \geqslant 2,\ k \geqslant 2. \tag{4.20}$$

GEOID HEIGHT RESULTS

Figure 1 presents a plot of the classical geoid height truncation error as a function of cap size based on equation (2.10). This graph shows that the expected error in geoid height does not decrease asymptotically as the area of integration is increased as might be expected intuitively. The error function has two local minima occurring at the zeros of the Stokes kernel. Even if the cap size is extended to sixty degrees, the truncation error still exceeds 10 meters.

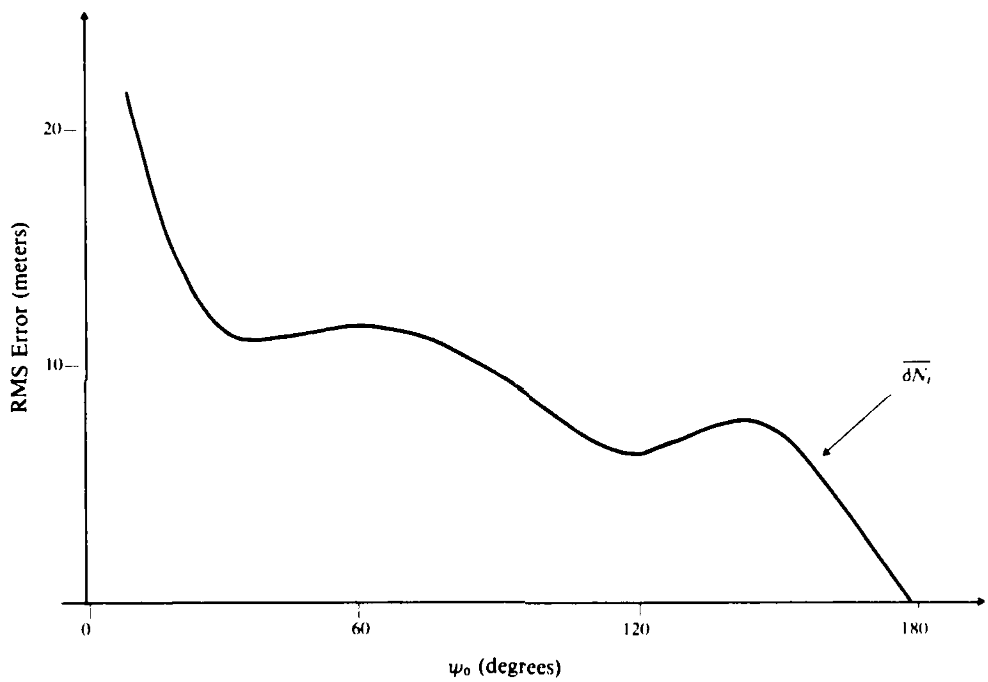


Figure 1. Truncation Error for Geoid Height

To reduce this error, two procedures were introduced with the resultant truncation errors $\overline{\delta N}_m$ [Fell, 1978] and $\overline{\delta N}_M$ [Molodenskii, 1962]. These procedures were evaluated for several values of M representing the number of harmonics of the gravity anomaly field which are assumed to be known. Figure 2 gives the resultant truncation errors for these procedures for the case where M is 6. In this case the truncation error was substantially reduced by the removal of 6 harmonics from the field ($\overline{\delta N}_m$) and dramatically reduced even further by the Molodenskii technique ($\overline{\delta N}_M$). As an example, for a cap size of twenty degrees the classical truncation error of 14.8 meters was reduced to 2.9 meters by the removal of 6 harmonics from the gravity anomaly field and reduced to 0.5 meters using the Molodenskii procedure which modifies the Stokes kernel according to equation (2.14). Figure 3 gives the modified and original Stokes kernel for this example, and Figure 4 presents the truncation errors $\overline{\delta N}_m$ and $\overline{\delta N}_M$ as a function of M for a twenty degree cap.

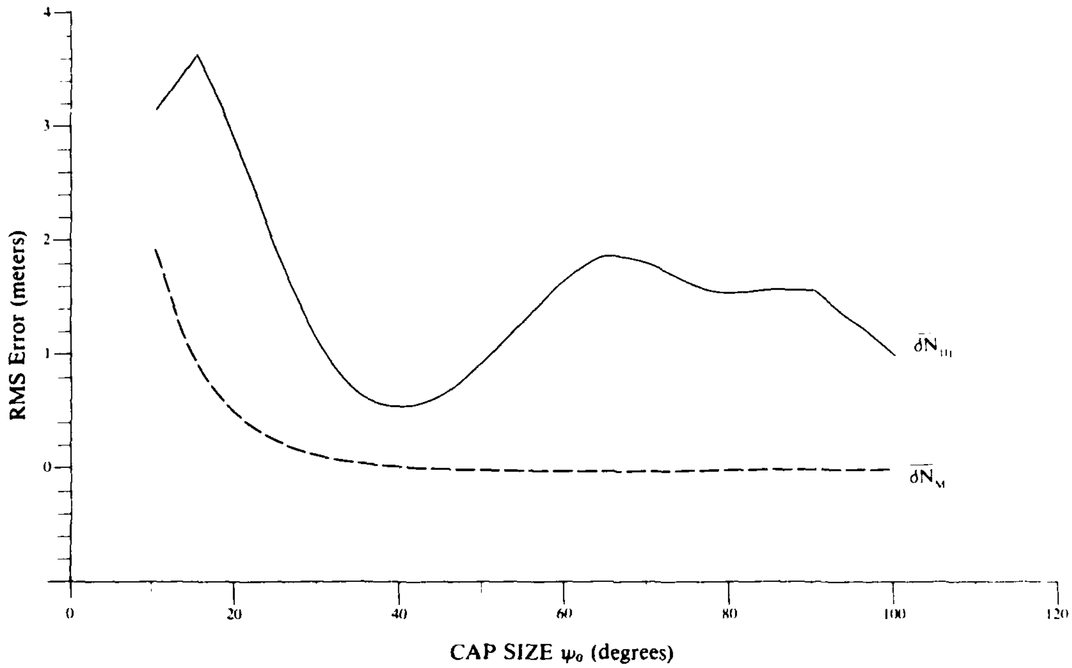


Figure 2. Expected Truncation Error for Geoid Height Using Method III and Molodenskii's Procedure ($M = 6$).

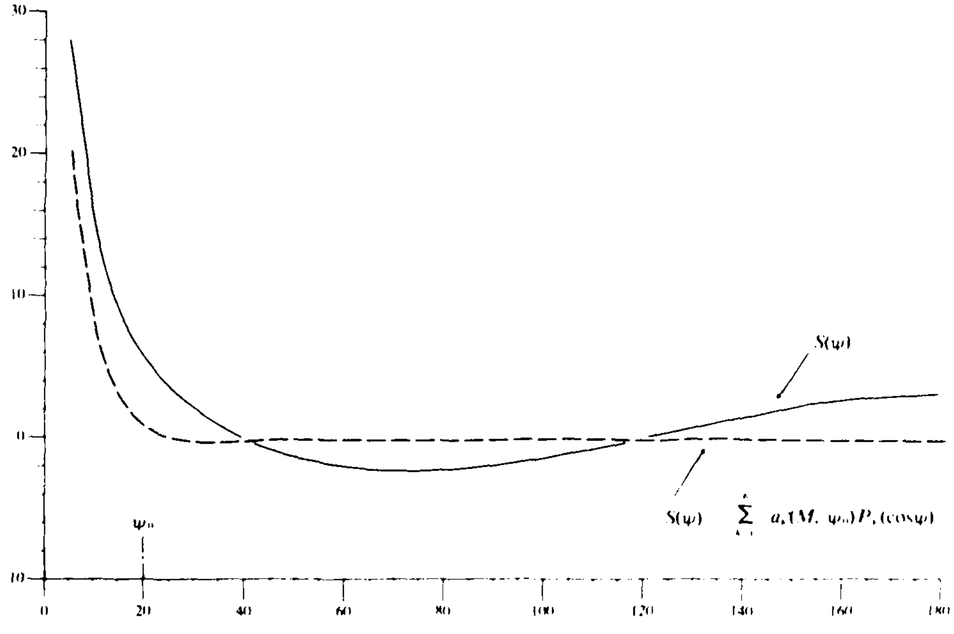


Figure 3. Stokes Function and Modified Kernel According to Molodenskii ($M = 6$, $\psi_o = 20°$).

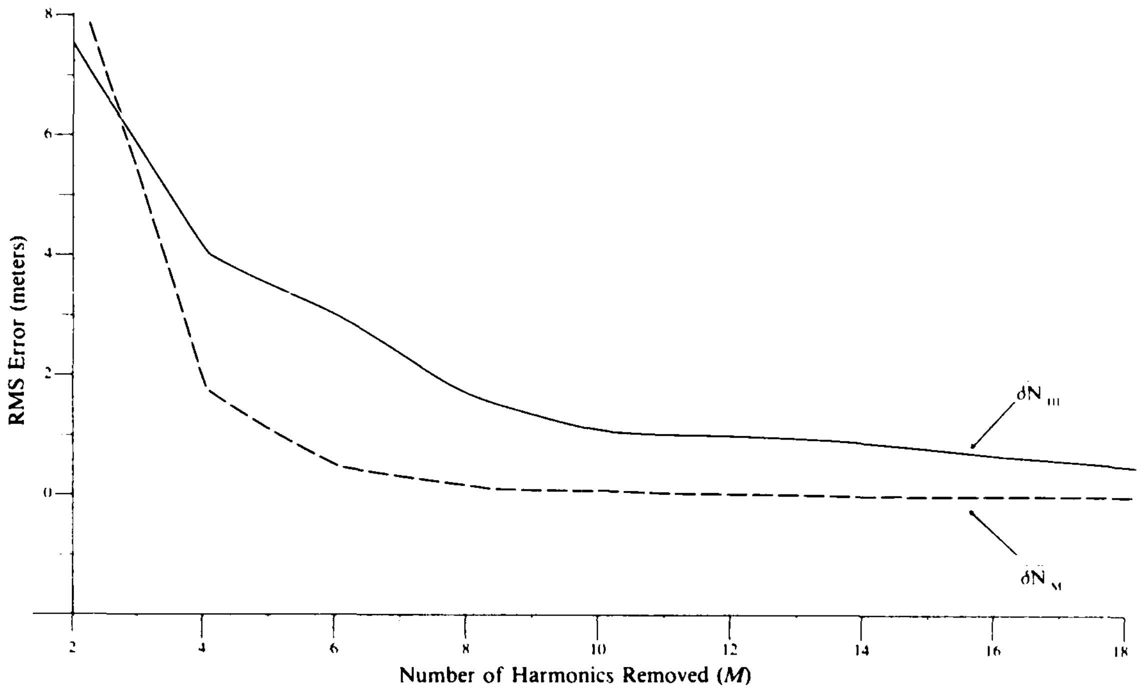


Figure 4. Truncation Error for Geoid Height as a Function of the Number of Known Harmonics (ψ_0 = 20°)

Appendix A contains additional information and results obtained using these two procedures for reducing the truncation error. The polynomial coefficients $a_k^0(M, \psi_0)$ for the kernel modifications using Molodenskii's procedure are given for several values of M. Examples of the modified Stokes kernel $S_M(\psi)$ are also plotted in this appendix. An examination of these figures shows that for increased values of M or for larger cap sizes the modified kernel according to Molodenskii approaches zero outside the cap as expected since this was the aim of the minimization criterion. Inside the cap the modified kernel appears somewhat like the result expected of a cosine taper applied to $S(\psi)$. Figures A.3 give additional examples of the truncation error as a function of cap size for M = 2, 4, 12 and Figures A.4 give the truncation errors $\overline{\delta N}_m$ and $\overline{\delta N}_M$ as a function of M for cap sizes of 15 and 30 degrees. Finally Tables A.2 give the truncation errors for M = 2, 4, 6, 8, 10 and 12 as a function of cap size ranging to 100 degrees. These results demonstrate for most cases that the Molodenskii technique is an extremely effective procedure for reducing the geoid height truncation error. The procedure examined by Fell (1978) consisting of harmonic removal from the anomaly data alone also significantly reduces the error of truncation but not nearly as well as the optimized procedure of modifying the kernel. The subtraction of the first M harmonics of the Stokes kernel with coefficients $(2n + 1)/(n - 1)$ from $S(\psi)$ is another (not optimized) procedure examined by Fell (1978) which cannot be expected to yield as good of results as with the Molodenskii technique. However the Molodenskii procedure is weakest when applied with small M and small cap size as evidenced in Tables A.2. In these cases the minimization procedure

is equivalent to fitting a low degree polynomial to the Stokes' kernel outside a small cap region. With a low order polynomial one cannot expect the difference between $S(\psi)$ and the polynomial to be small in these cases after minimization. Recall that $S(\psi)$ requires an infinte series of Legendre polynomials for an expansion over the interval of definition. In these cases the procedure works but not as effectively as in other cases.

VERTICAL DEFLECTION RESULTS

The truncation error for vertical deflection based on equation (3.13) is given in Figure 5 as a function of cap size. For instance, using observations within a forty degree cap would result in an rms truncation error of approximately 1 second of arc.

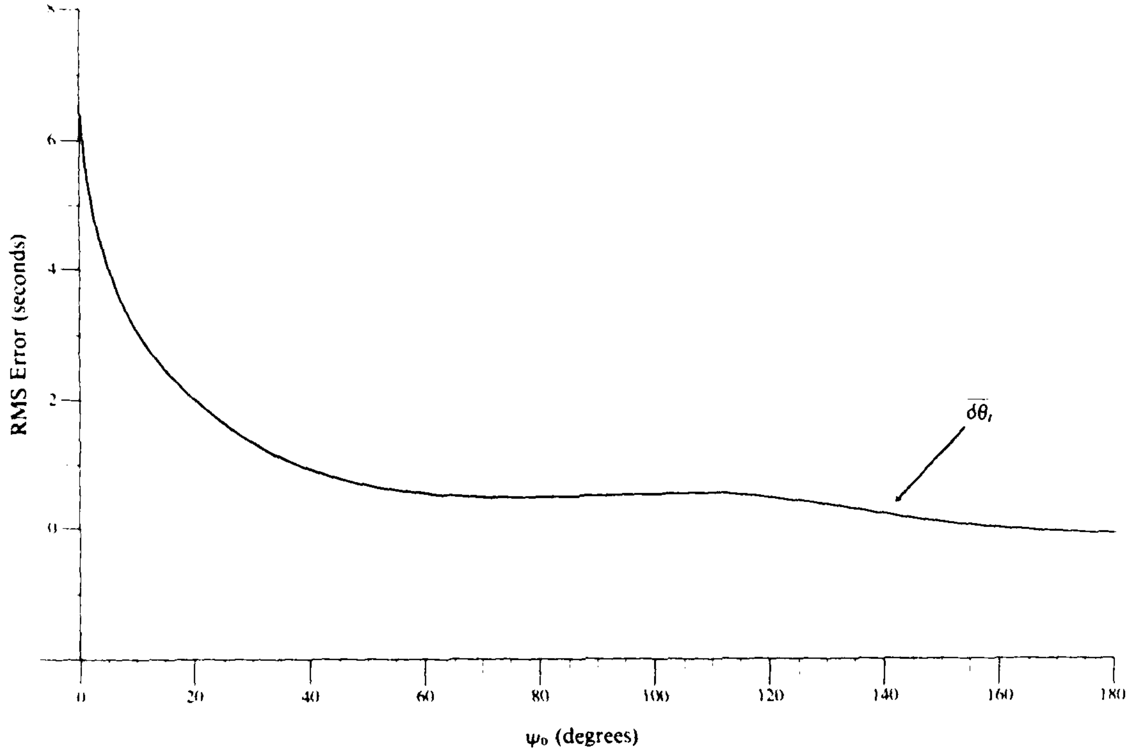


Figure 5. Truncation Error for Vertical Deflection

To reduce this error two procedures are examined. The first is based on removal of harmonics from the gravity anomaly field adopting the Vening Meinesz kernel, while the second additionally modifies the kernel according to equation (3.14). In this latter approach the Molodenskii technique for geoid height is modified by changing the form of the approximating polynomial. Figures 6 and 7 give examples of this modified kernel for M equal to 6 for two cap sizes, 20 and 40 degrees. For the latter case the modified kernel resembles a cosine taper applied to the Vening Meinesz kernel. In Figure 8 the expected truncation error for vertical deflection as a function of cap size is illustrated for the three cases mentioned. Substantial reduction in the truncation error is apparent using Method III and the Molodenskii technique, with the latter producing better results for caps sizes exceeding 20 degrees. In this example the Molodenskii procedure would indicate that a cap size of between 20 and 30 degrees would be adequate for a deflection accuracy of 0.2 seconds of arc. This information is thus valuable for survey design. For a desired accuracy and for an assumed value for M a minimum cap size may be determined from graphs analogous to Figure 8. If deflections are desired in a limited geographical region, then the extent of the survey to support deflection computations in that region may be defined.

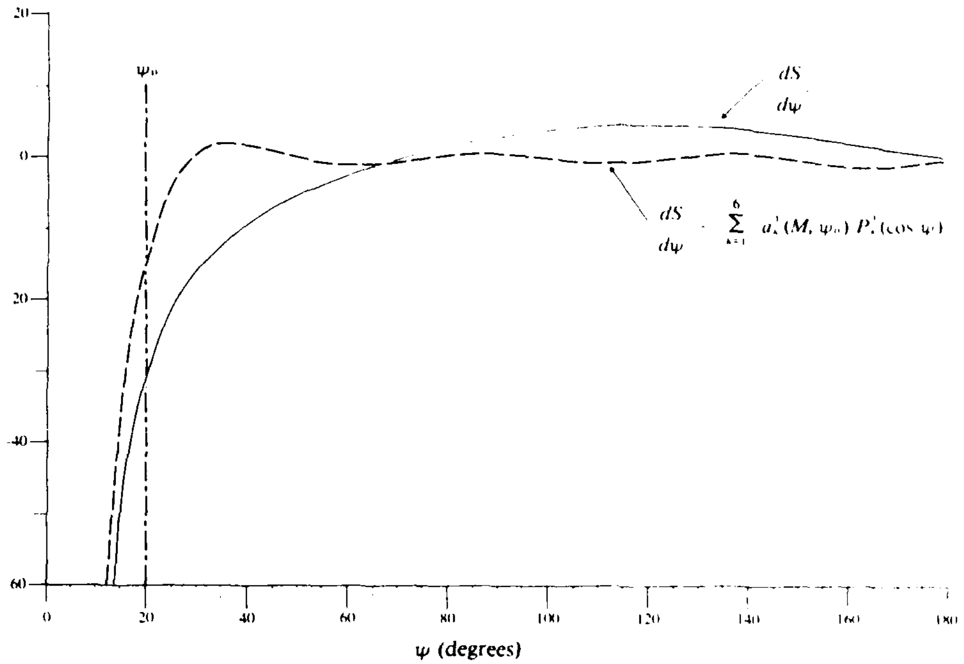


Figure 6. Vening Meinesz Kernel and Modified Kernel Based on a Molodenskii Approach ($M = 6$, $\psi_0 = 20°$)

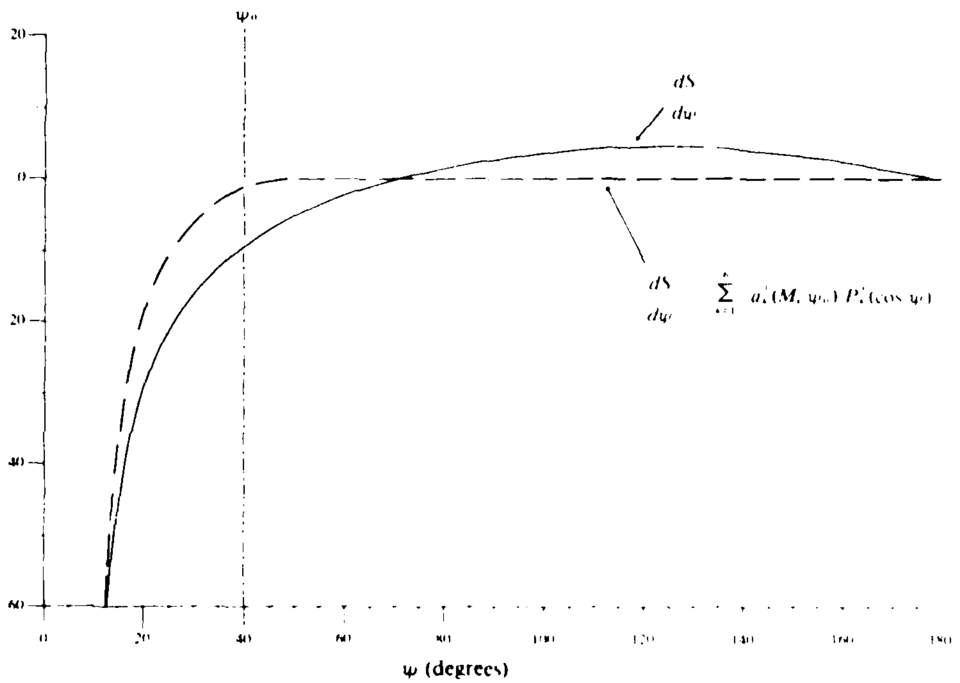


Figure 7. Vening Meinesz Kernel and Modified Kernel Based on a Molodenskii Approach ($M = 6$, $\psi_0 = 40°$)

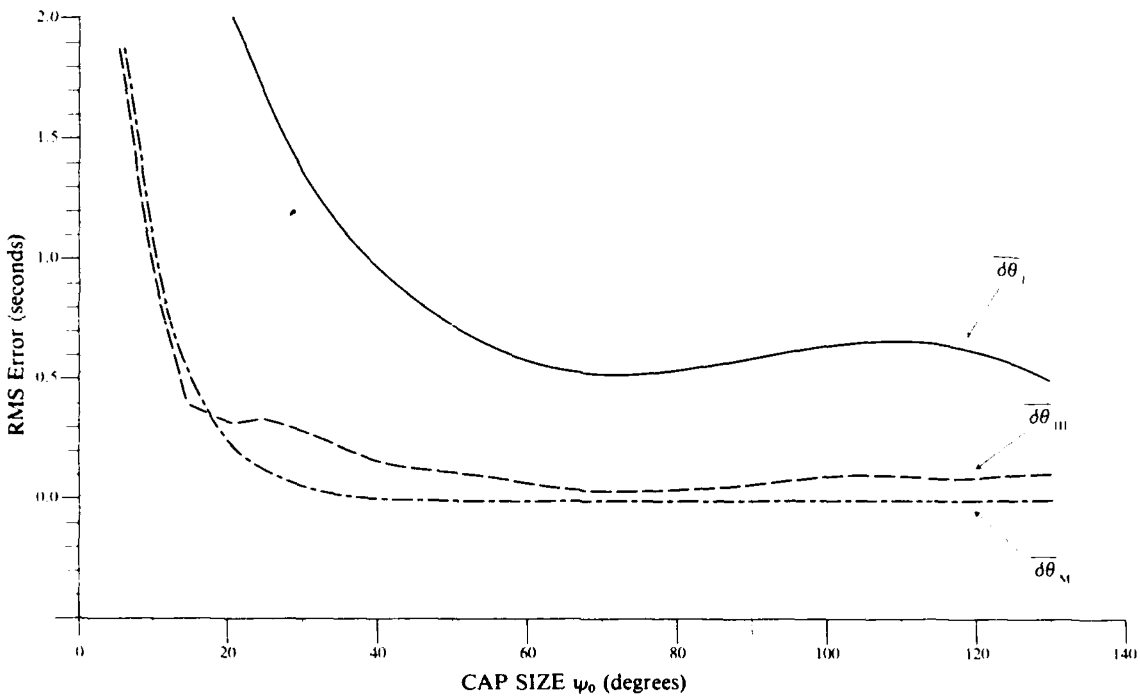


Figure 8. Expected Truncation Error for Vertical Deflection as a Function of Cap Size (M = 6).

Figure 9 gives an example of the rms deflection error for a 30 degree cap as a function of the number of harmonics assumed known. Again for small caps and small M the approximation of the Vening Meinesz kernel with the low degree polynomial will not be as effective a procedure, since the modified kernel may be significantly different from zero exterior to the cap.

In Appendix B more detailed results are presented for these two procedures for minimizing the truncation error. Table B.1 gives the coefficients $a_k^1(M, \psi_0)$ for the modified Vening Meinesz kernel based on the Molodenskii procedure for M equal to 2, 4, 6, 8 and 10. Figures B.1 and B.2 give examples of these modified kernels for cap sizes of 20 and 40 degrees respectively. Figure B.4 gives the expected truncation error for vertical deflection using Method III and the Molodenskii approach as a function of the number of harmonics removed for the 20 and 40 degree spherical caps. And finally, in Table B.2 the truncation errors $\overline{\delta\theta}_I$, $\overline{\delta\theta}_{III}$ and $\overline{\delta\theta}_M$ are given for cap sizes up to 100 degrees for M values of 2, 4, 6, 8, 10, and 12. Again for small values of M and ψ_0 these tables show that the Molodenskii approach and Method III give comparable results. As the number of harmonics removed and the cap size increases the Molodenskii approach becomes superior.

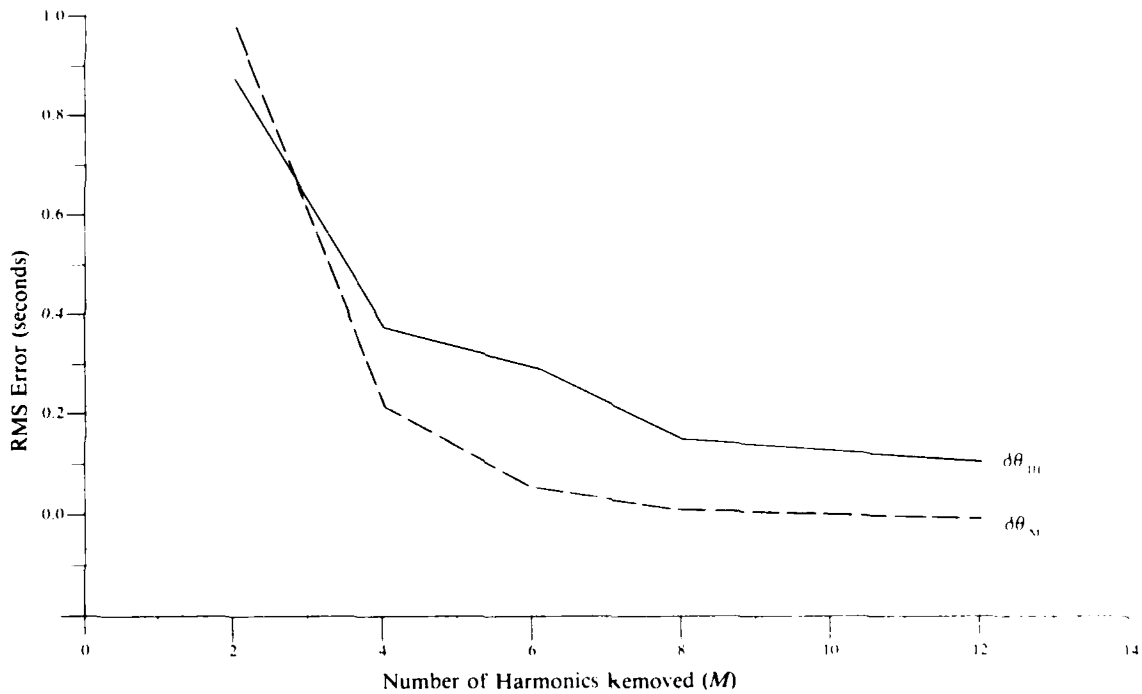


Figure 9. Expected Truncation Error for Vertical Deflection as a Function of the Number of Known Harmonics ($\psi_0 = 30°$)

CONCLUSIONS

This report has addressed some relatively simple approaches for minimizing truncation error for geoid height and deflection of the vertical computations due to the application of the Stokes and Vening Meinesz integrals on limited geographic areas. The results clearly demonstrate that major reductions in these truncation errors are possible under the assumption that the lower degree harmonics of the disturbing potential are known. The effects of errors in these harmonics and errors in gravity anomaly data were not considered.

REFERENCES

Cooke, A. H., 1950. "The Ca: ulation of Deflexions of the Vertical From Gravity Anomalies," *Proc. Roy. Soc., A*, 204, 374-395.

deWitte, L., 1967. "Truncation Errors in the Stokes and Vening Meinesz Formulae for Different Order Spherical Harmonic Gravity Terms," *Geophys. J. R. Astr. Soc.*, 12, 449-464.

Dickson, R. J., 1979. *Truncation Error in the Problem of Stokes*, Lockheed Palo Alto Research Laboratory, TM/81-15/61.

Fell, P. J., 1978. *The Influence of Distant Zones on Stokes' Equation Considering the Removal of Lower-Degree Harmonics from* $S(\psi)$ or Δg, NSWC/DL TR-3911, Dahlgren, Virginia.

Hagiwara, Y., 1973. "Truncation Error Formulas for the Geoidal Height and the Deflection of the Vertical," *Bulletin Geodesique*, 106, 453-466.

Heiskanen, W. and Moritz, H., 1967. *Physical Geodesy*, W. H. Freeman, San Francisco, CA.

Lebebev, N. N., 1972. *Special Functions and Application*, Dover Publications, Inc., New York, NY.

Meissl, P., 1971. "Preparations for the Numerical Evaluation of Second Order Molodenskii-Type Formulas," Dept. of Geodetic Science Report 163, The Ohio State University, Columbus, Ohio.

Molodenskii, M. S., Eremeev, V. F., and Urkina, M. I., 1962. *Methods for Study of the External Gravitational Field and Figure of the Earth*, Translated from the Russian by the Israel Program for Scientific Translation, Jerusalem.

Paul, M K., 1973. "A Method of Evaluating the Truncation Error Coefficients for Geoidal Height," *Bulletin Geodesique*, 110, 413-425.

Wong, L, and Gore, R., 1969. "Accuracy of Geoid Heights from Modified Stokes Kernels," *Geophys, J. R. Astr. Soc.*, 18, 81-91.

APPENDIX A

MODIFIED STOKES KERNELS AND TRUNCATION ERRORS

Table A.1 Polynomial Coefficients $a_k^0(M, \psi_0)$ for Stokes Kernel Modifications According to Molodenskii

$M = 2$

ψ_0	a_0^0	a_1^0	a_2^0
10.0	-.18183	-.54283	4.10415
15.0	-.26676	-.79184	3.70801
20.0	-.34709	-1.02267	3.35591
25.0	-.42365	-1.23737	3.04532
30.0	-.49728	-1.43820	2.77224
35.0	-.56865	-1.62715	2.53235
40.0	-.63827	-1.80583	2.32154
45.0	-.70647	-1.97549	2.13608
50.0	-.77350	-2.13706	1.97267
55.0	-.83947	-2.29123	1.82843
60.0	-.90444	-2.43849	1.70088

$M = 4$

ψ_0	a_0^0	a_1^0	a_2^0	a_3^0	a_4^0
10.0	-.14969	-.44711	4.26130	2.47930	1.71041
15.0	-.20709	-.61574	3.99193	2.12596	1.29522
20.0	-.25791	-.76264	3.76498	1.84379	.98873
25.0	-.30444	-.89478	3.56835	1.61391	.76178
30.0	-.34805	-1.01639	3.39429	1.42333	.59270
35.0	-.38956	-1.13005	3.23785	1.26312	.46579
40.0	-.42947	-1.23736	3.09575	1.12696	.36975
45.0	-.46807	-1.33933	2.96575	1.01028	.29645
50.0	-.50555	-1.43662	2.84625	.90968	.24004
55.0	-.54200	-1.52965	2.73605	.82250	.19626
60.0	-.57749	-1.61872	2.63423	.74668	.16200

$M = 6$

ψ_0	a_0^0	a_1^0	a_2^0	a_3^0	a_4^0	a_5^0	a_6^0
10.0	-.13010	-.38876	4.35713	2.61052	1.87419	1.40071	1.04240
15.0	-.17431	-.51892	4.14827	2.33460	1.54658	1.03965	.66796
20.0	-.21263	-.63034	3.97408	2.11371	1.29918	.78807	.43449
25.0	-.24742	-.73017	3.82216	1.92909	1.10486	.60724	.28720
30.0	-.27985	-.82203	3.68603	1.77055	.94808	.47404	.19298
35.0	-.31056	-.90792	3.56205	1.63206	.81930	.37409	.13182
40.0	-.33990	-.98899	3.44799	1.50977	.71223	.29802	.09152
45.0	-.36812	-1.06598	3.34238	1.40099	.62243	.23950	.06456
50.0	-.39533	-1.13936	3.24419	1.30375	.54664	.19405	.04627
55.0	-.42164	-1.20946	3.15264	1.21652	.48236	.15850	.03367
60.0	-.44709	-1.27649	3.06716	1.13808	.42761	.13049	.02488

Table A.1 (Continued)

$M = 8$

ψ_0	a_0^0	a_1^0	a_2^0	a_3^0	a_4^0	a_5^0	a_6^0	a_7^0	a_8^0
10.0	-.11624	-.34748	4.42497	2.70348	1.99031	1.53757	1.19716	.92073	.68826
15.0	-.15273	-.45515	4.25146	2.47272	1.71368	1.22864	.87095	.59298	.37456
20.0	-.18421	-.54707	4.10635	2.28578	1.49933	1.00306	.65064	.39329	.20838
25.0	-.21271	-.62951	3.97884	2.12664	1.32482	.83013	.49480	.26672	.11861
30.0	-.23922	-.70539	3.86380	1.98747	1.17874	.69368	.38122	.18409	.06908
35.0	-.26422	-.77629	3.75842	1.86382	1.05441	.58408	.29681	.12897	.04116
40.0	-.28802	-.84314	3.66096	1.75287	.94746	.49500	.23321	.09156	.02508
45.0	-.31080	-.90653	3.57030	1.65265	.85477	.42197	.18479	.06583	.01562
50.0	-.33268	-.96686	3.48562	1.56173	.77403	.36172	.14760	.04791	.00994
55.0	-.35375	-1.02441	3.40631	1.47898	.70341	.31174	.11884	.03528	.00646
60.0	-.37404	-1.07937	3.33194	1.40350	.64144	.27009	.09644	.02630	.00428

$M = 10$

ψ_0	a_0^0	a_1^0	a_2^0	a_3^0	a_4^0	a_5^0	a_6^0	a_7^0	a_8^0	a_9^0	a_{10}^0
10.0	-.10578	-.31631	4.47621	2.77375	2.07820	1.64130	1.31465	1.04966	.82610	.63422	.46880
15.0	-.13725	-.40934	4.32573	2.57244	1.83486	1.36648	1.02012	.74783	.52944	.35464	.21709
20.0	-.16438	-.48887	4.19922	2.40748	1.64237	1.15888	.81033	.54823	.35089	.20571	.10349
25.0	-.18893	-.56025	4.08752	2.26545	1.48228	.99390	.65301	.40919	.23778	.12263	.05082
30.0	-.21169	-.62592	3.98639	2.13999	1.34561	.85918	53165	.30940	.16382	.07471	.02570
35.0	-.23311	-.68722	3.89350	2.02752	1.22712	.74738	.43639	.23642	.11442	.04638	.01338
40.0	-.25343	-.74493	3.80738	1.92573	1.12336	.65360	.36074	.18233	.08091	.02929	.00717
45.0	-.27282	-.79958	3.72708	1.83302	1.03188	.57435	.30015	.14184	.05788	.01881	.00395
50.0	-.29138	-.85153	3.65191	1.74823	.95082	.50698	.25128	.11126	.04188	.01227	.00223
55.0	-.30921	-.90102	3.58135	1.67043	.87875	.44945	.21163	.08799	.03064	.00814	.00130
60.0	-.32633	-.94823	3.51501	1.59891	.81448	.40014	.17931	.07017	.02268	.00549	.00077

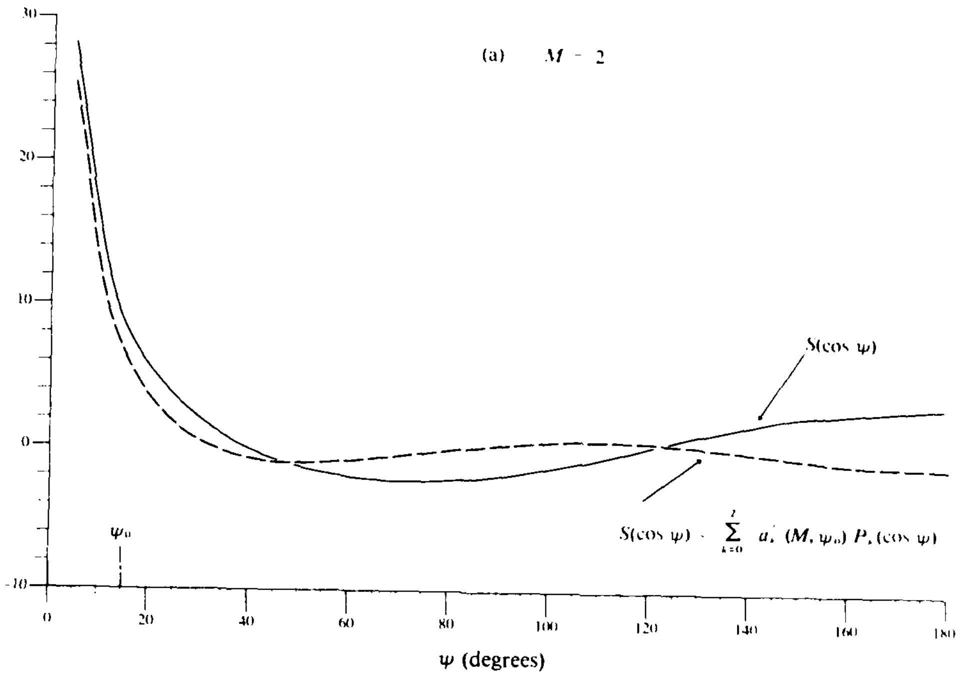


Figure A.1 Stokes Function and Modified Kernel According to Molodenskii (ψ_0 = 15°)

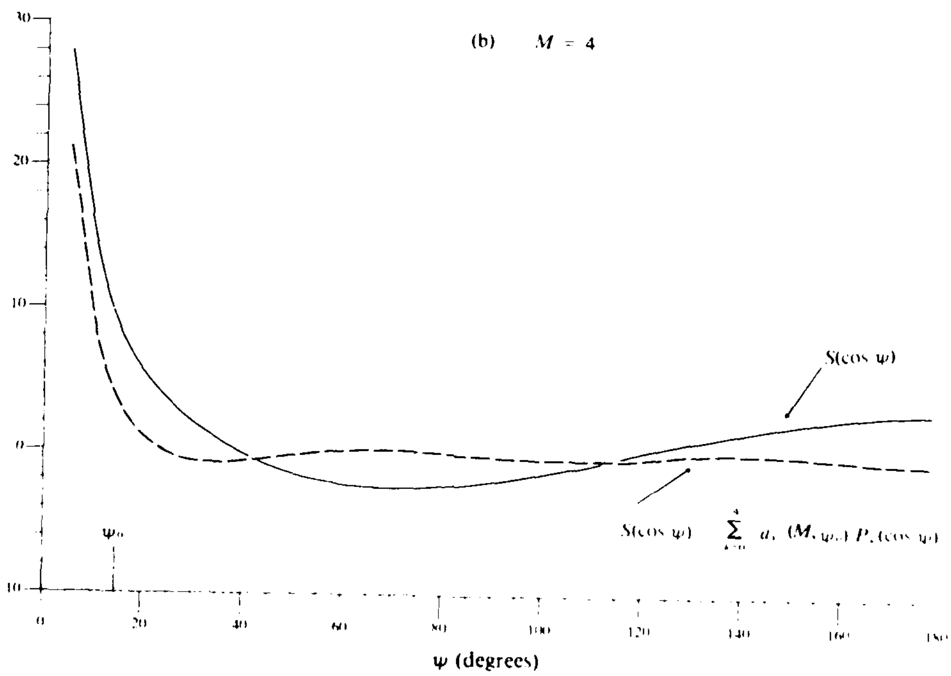


Figure A.1 Stokes Function and Modified Kernel According to Molodenskii (ψ_0 = 15°)

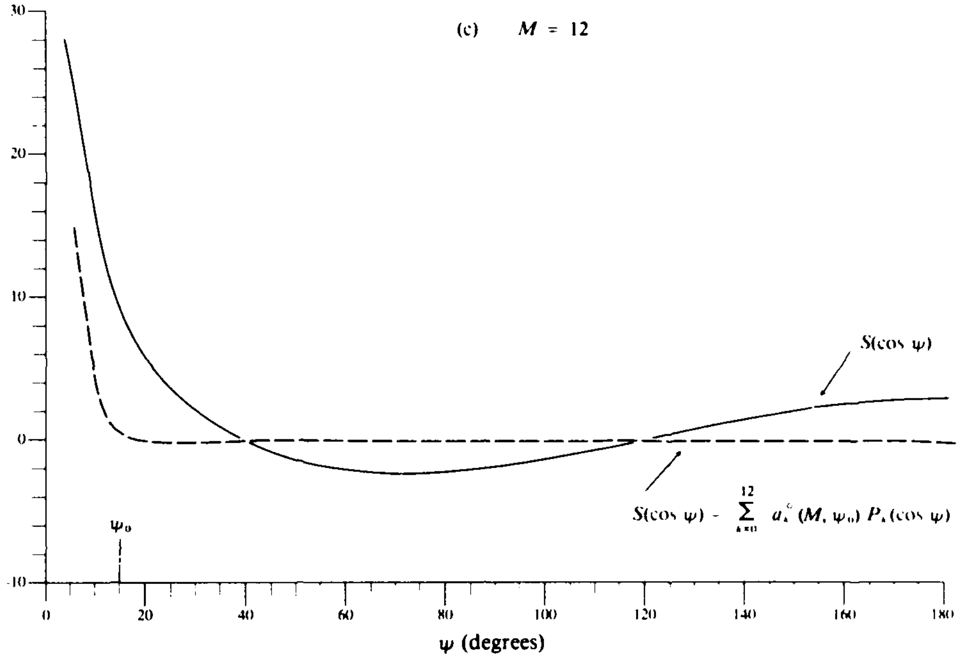


Figure A.1 Stokes Function and Modified Kernel According to Molodenskii (ψ_0 = 15°)

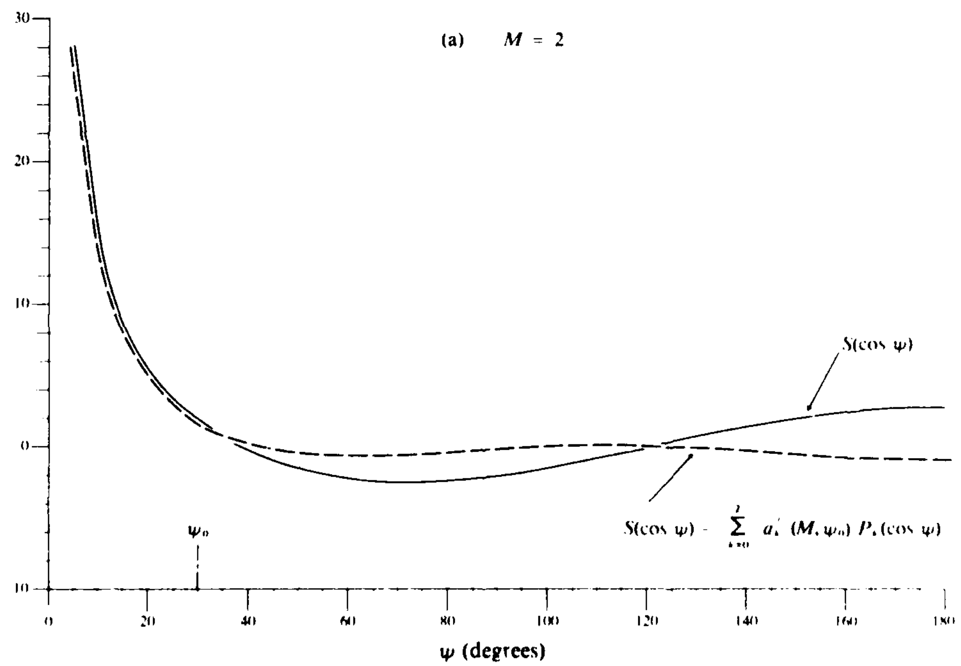


Figure A.2 Stokes Function and Modified Kernel According to Molodenskii (ψ_0 = 30°)

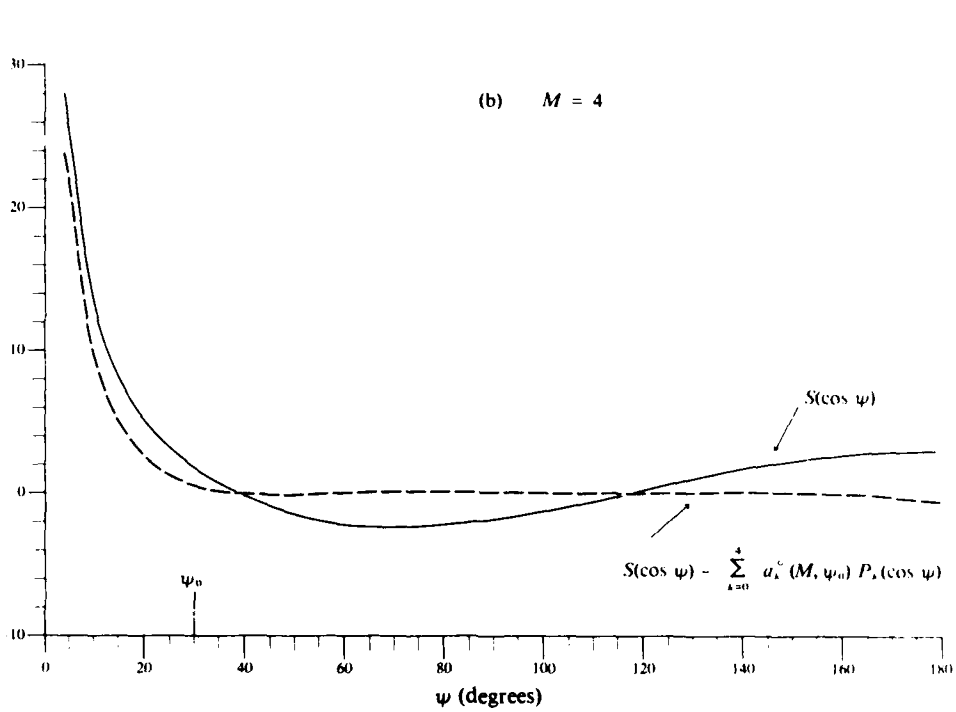


Figure A.2 Stokes Function and Modified Kernel According to Molodenskii ($\psi_0 = 30°$)

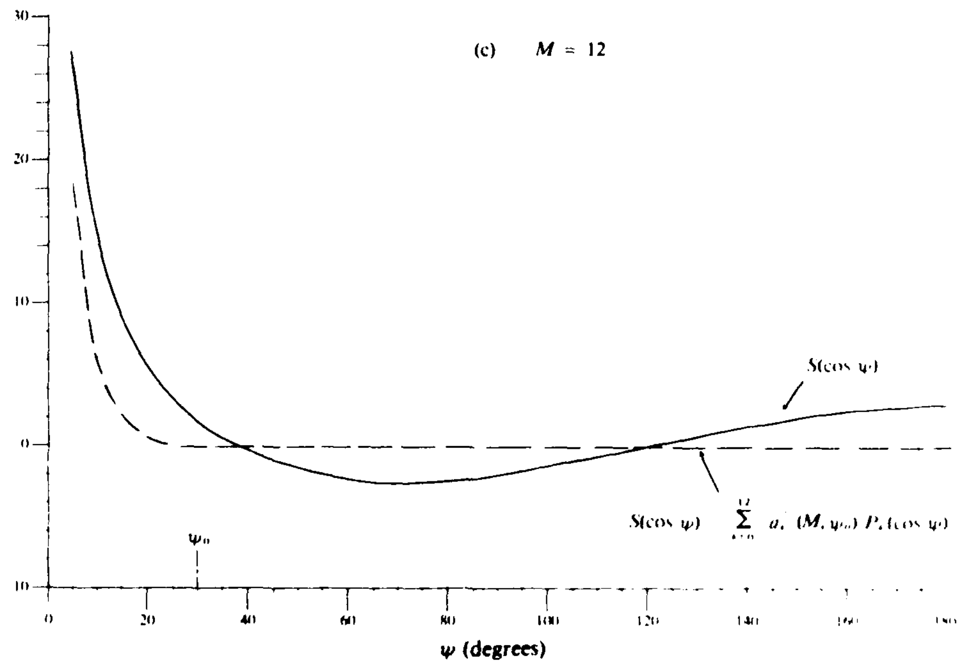


Figure A.2 Stokes Function and Modified Kernel According to Molodenskii ($\psi_0 = 30°$)

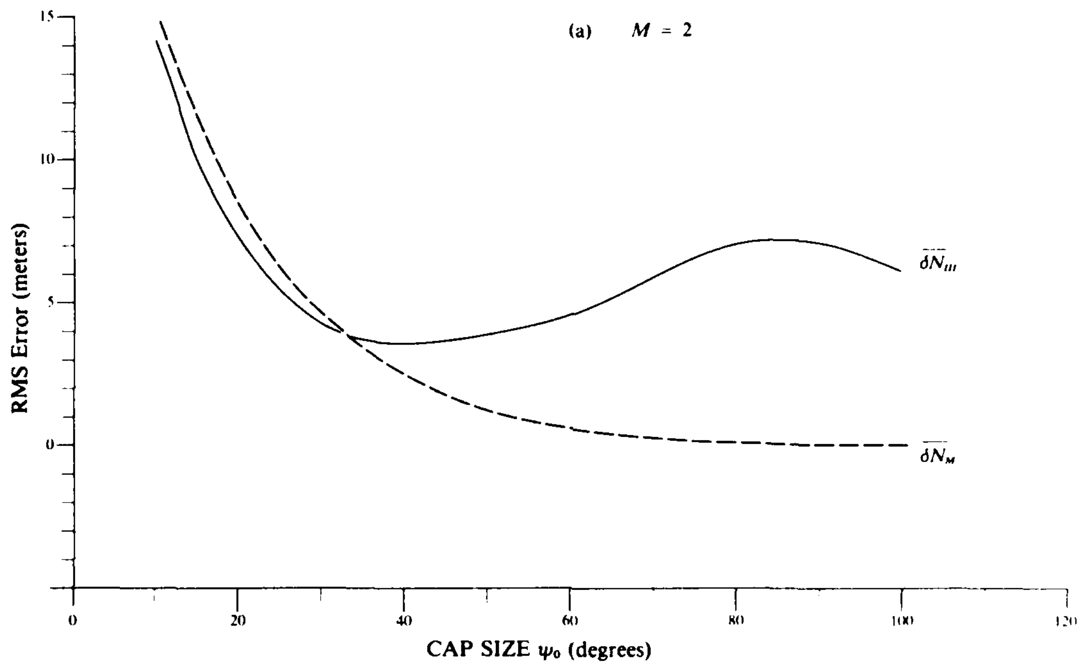


Figure A.3 Expected Truncation Error (RMS) for Geoid Height using Method III and Molodenskii's Procedure.

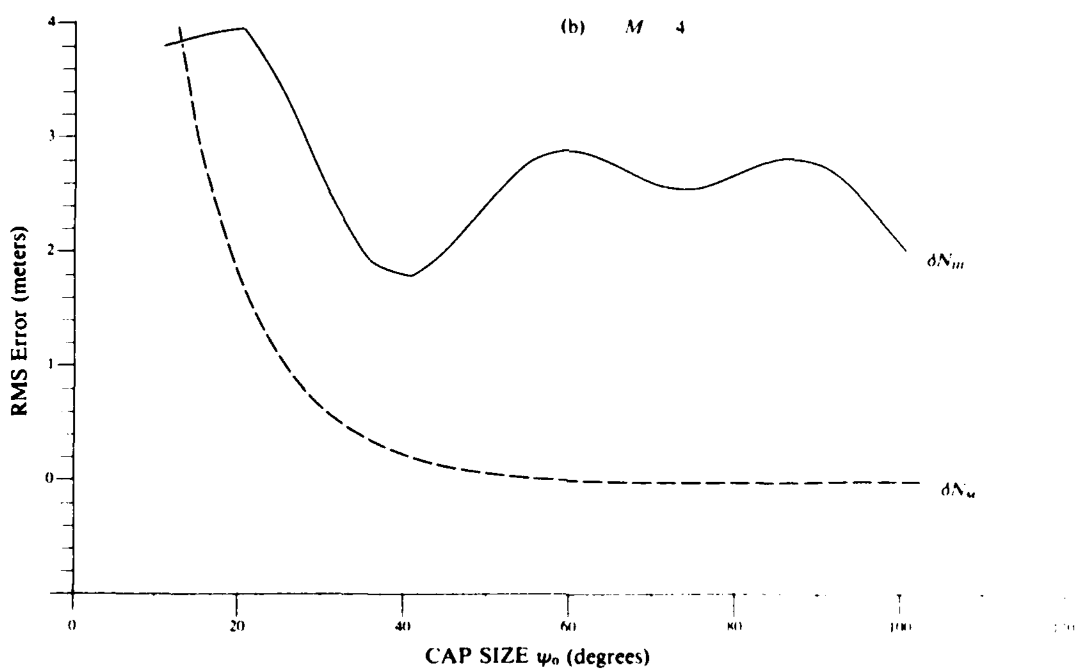


Figure A.3 Expected Truncation Error (RMS) for Geoid Height using Method III and Molodenskii's Procedure.

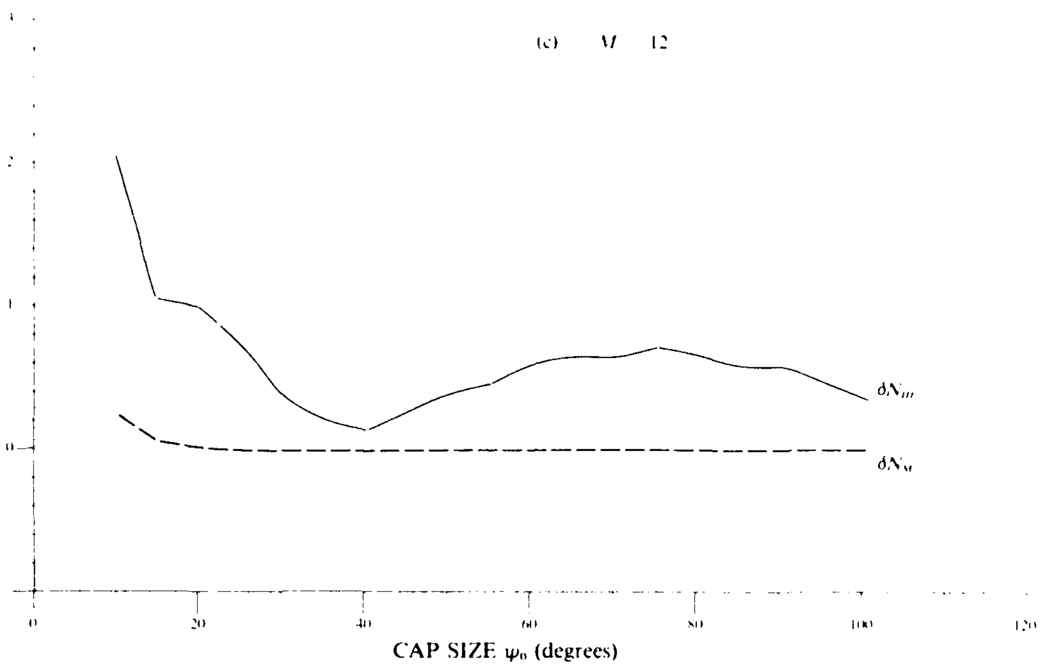


Figure A.3 Expected Truncation Error (RMS) for Geoid Height using Method III and Molodenskii's Procedure.

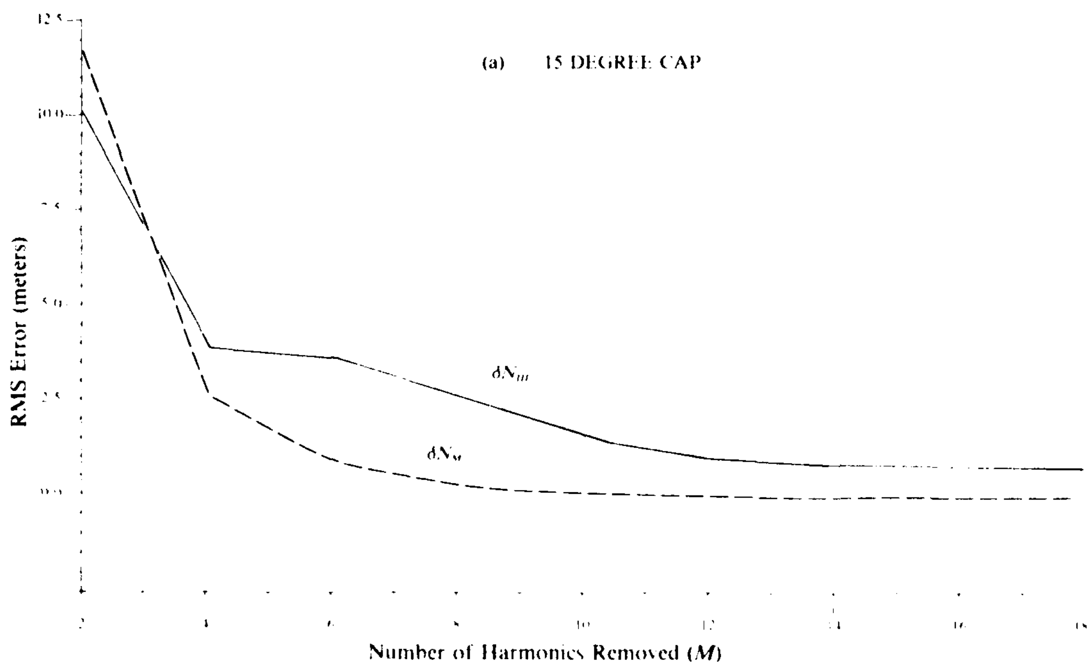


Figure A.4 Expected Truncation Error (RMS) for Geoid Height using Method III and Molodenskii's Procedure as a Function of the Number of Harmonics Removed.

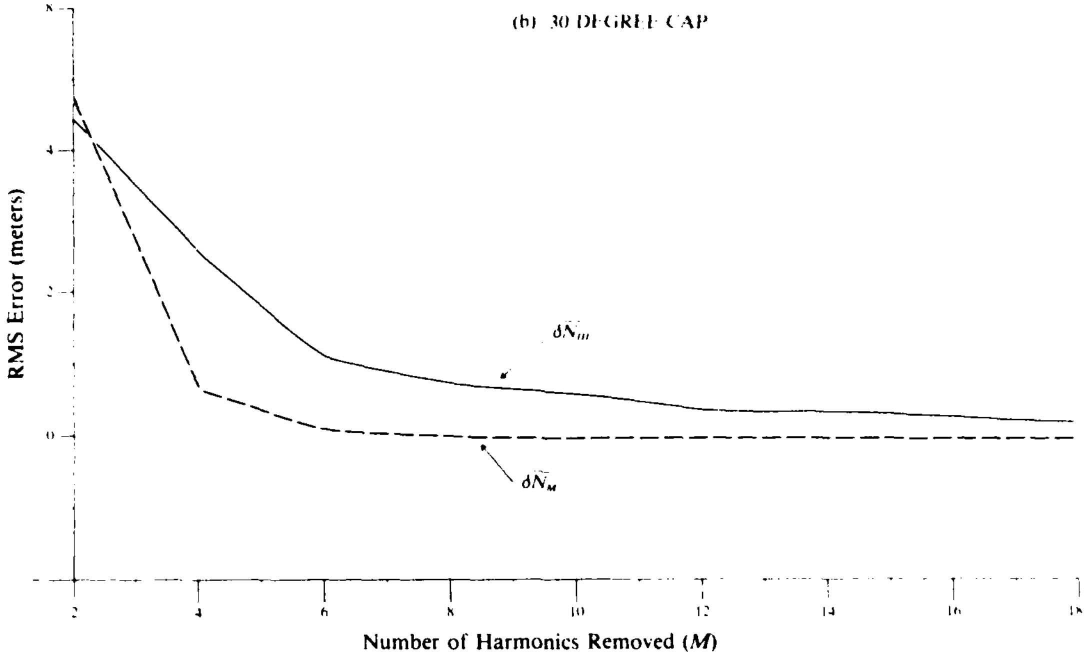


Figure A.4 Expected Truncation Error (RMS) for Geoid Height using Method III and Molodenskii's Procedure as a Function of the Number of Harmonics Removed.

Table A.2 Expected Truncation Error (meters) for Geoid Height Based on Method III and Molodenskii's Procedure

M = 2

ψ_0	$\overline{\delta N}_I$	$\overline{\delta N}_{III}$	$\overline{\delta N}_M$
5.0	27.669661	20.556343	21.107032
10.0	21.739670	14.294292	15.528263
15.0	17.587212	10.057129	11.502553
20.0	14.777943	7.337731	8.555670
25.0	12.916628	5.565207	6.373636
30.0	11.775998	4.406253	4.744508
35.0	11.229530	3.798171	3.521937
40.0	11.127562	3.681122	2.602423
45.0	11.269953	3.830107	1.911084
50.0	11.471612	4.043541	1.392643
55.0	11.619539	4.296507	1.005635
60.0	11.675333	4.682291	.718571
65.0	11.641289	5.260989	.507344
70.0	11.521156	5.965933	.353411
75.0	11.298368	6.642020	.242489
80.0	10.939754	7.131865	.163589
85.0	10.416799	7.330423	.108290
90.0	9.728997	7.204085	.070178
95.0	8.917664	6.789355	.044406
100.0	8.066346	6.180600	.027350

M = 4

ψ_0	$\overline{\delta N}_I$	$\overline{\delta N}_{III}$	$\overline{\delta N}_M$
5.0	27.669661	7.143403	8.144118
10.0	21.739670	3.821151	4.687022
15.0	17.587212	3.916332	2.780762
20.0	14.777943	3.972130	1.678309
25.0	12.916628	3.379162	1.021338
30.0	11.775998	2.561030	.622904
35.0	11.229530	1.940179	.379109
40.0	11.127562	1.785319	.229512
45.0	11.269953	2.072378	.137855
50.0	11.471612	2.514934	.081967
55.0	11.619539	2.845642	.048144
60.0	11.675333	2.924490	.027876
65.0	11.641289	2.773443	.015878
70.0	11.521156	2.576939	.008877
75.0	11.298368	2.556387	.004859
80.0	10.939754	2.719752	.002597
85.0	10.416799	2.859411	.001352
90.0	9.728997	2.797542	.000682
95.0	8.917664	2.495368	.000333
100.0	8.066346	2.027741	.000156

Table A.2 (Continued)

$M = 6$

ψ_0	$\overline{\delta N}_I$	$\overline{\delta N}_{III}$	$\overline{\delta N}_{VI}$
5.0	27.669661	3.264089	4.202758
10.0	21.739670	3.167549	1.941453
15.0	17.587212	3.645828	.943511
20.0	14.777943	2.936724	.470071
25.0	12.916628	1.892564	.236672
30.0	11.775998	1.093629	.119415
35.0	11.229530	.640365	.060057
40.0	11.127562	.530456	.029990
45.0	11.269953	.685078	.014823
50.0	11.471612	.936809	.007231
55.0	11.619539	1.308044	.003473
60.0	11.675333	1.698420	.001638
65.0	11.641289	1.900181	.000756
70.0	11.521156	1.826413	.000341
75.0	11.298368	1.622609	.000149
80.0	10.939754	1.551919	.000064
85.0	10.416799	1.620535	.000026
90.0	9.728997	1.592219	.000010
95.0	8.917664	1.361978	.000004
100.0	8.066346	1.016394	.000001

$M = 8$

ψ_0	$\overline{\delta N}_I$	$\overline{\delta N}_{III}$	$\overline{\delta N}_{VI}$
5.0	27.669661	1.992536	2.456855
10.0	21.739670	3.037807	.924879
15.0	17.587212	2.681047	.371975
20.0	14.777943	1.600779	.153939
25.0	12.916628	.988105	.064393
30.0	11.775998	.747235	.026968
35.0	11.229530	.478502	.011241
40.0	11.127562	.380003	.004643
45.0	11.269953	.570574	.001893
50.0	11.471612	.794426	.000760
55.0	11.619539	.883838	.000299
60.0	11.675333	.950892	.000115
65.0	11.641289	1.161459	.000043
70.0	11.521156	1.309464	.000016
75.0	11.298368	1.223234	.000006
80.0	10.939754	1.072946	.000002
85.0	10.416799	1.075179	.000001
90.0	9.728997	1.062328	.000000
95.0	8.917664	.879693	.000000
100.0	8.066346	.639448	.000000

Table A.2 (Continued)

$M = 10$

ψ_0	$\overline{\delta N}_I$	$\overline{\delta N}_{III}$	$\overline{\delta N}_M$
5.0	27.669661	1.730302	1.542910
10.0	21.739670	2.608021	.477840
15.0	17.587212	1.696188	.159824
20.0	14.777943	1.042413	.055085
25.0	12.916628	.944152	.019181
30.0	11.775998	.615120	.006679
35.0	11.229530	.296121	.002311
40.0	11.127562	.206595	.000791
45.0	11.269953	.317089	.000266
50.0	11.471612	.486039	.000088
55.0	11.619539	.709787	.000028
60.0	11.675333	.798808	.000009
65.0	11.641289	.784758	.000003
70.0	11.521156	.899044	.000001
75.0	11.298368	.949006	.000000
80.0	10.939754	.825826	.000000
85.0	10.416799	.780515	.000000
90.0	9.728997	.773971	.000000
95.0	8.917664	.625620	.000000
100.0	8.066346	.470106	.000000

$M = 12$

ψ_0	$\overline{\delta N}_I$	$\overline{\delta N}_{III}$	$\overline{\delta N}_M$
5.0	27.669661	1.716455	1.015893
10.0	21.739670	2.058038	.260410
15.0	17.587212	1.086172	.072612
20.0	14.777943	1.016599	.020872
25.0	12.916628	.737799	.006057
30.0	11.775998	.381679	.001756
35.0	11.229530	.208457	.000505
40.0	11.127562	.141440	.000143
45.0	11.269953	.279372	.000040
50.0	11.471612	.410834	.000011
55.0	11.619539	.474923	.000003
60.0	11.675333	.624421	.000001
65.0	11.641289	.672848	.000000
70.0	11.521156	.650156	.000000
75.0	11.298368	.728428	.000000
80.0	10.939754	.671636	.000000
85.0	10.416799	.601482	.000000
90.0	9.728997	.596455	.000000
95.0	8.917664	.474451	.000000
100.0	8.066346	.378833	.000000

APPENDIX B

MODIFIED VENING MEINESZ KERNELS AND TRUNCATION ERRORS

TABLE B.1 Coefficients $a_k^1(M, \psi_0)$ for Modified Vening Meinesz Kernels Based on a Molodenskii Type Procedure.

$M = 2$

ψ_0	a_1^1	a_2^1
10.0	.29863	-4.50496
15.0	.47429	-4.21927
20.0	.66262	-3.91999
25.0	.85842	-3.61841
30.0	1.05679	-3.32462
35.0	1.25385	-3.04617
40.0	1.44691	-2.78780
45.0	1.63430	-2.55183
50.0	1.81517	-2.33875
55.0	1.98913	-2.14784
60.0	2.15614	-1.97770

$M = 4$

ψ_0	a_1^1	a_2^1	a_3^1	a_4^1
10.0	.29225	-4.51548	-2.82704	-2.14386
15.0	.44868	-4.26098	-2.48351	-1.72343
20.0	.60025	-4.01985	-2.16908	-1.35669
25.0	.74279	-3.79950	-1.89448	-1.05657
30.0	.87555	-3.60103	-1.66017	- .82044
35.0	.99943	-3.42251	-1.46178	- .63854
40.0	1.11574	-3.26117	-1.29365	- .49974
45.0	1.22569	-3.11445	-1.15052	- .39405
50.0	1.33022	-2.98025	-1.02798	- .31339
55.0	1.42998	-2.85693	- .92253	- .25155
60.0	1.52544	-2.74326	- .83136	- .20383

$M = 6$

ψ_0	a_1^1	a_2^1	a_3^1	a_4^1	a_5^1	a_6^1
10.0	.28330	-4.53025	-2.84741	-2.16952	-1.74788	-1.43369
15.0	.41916	-4.30910	-2.54864	-1.80345	-1.32703	-.97226
20.0	.54078	-4.11517	-2.29487	-1.50595	-1.00433	-.64394
25.0	.64954	-3.94597	-2.08179	-1.26934	-.76599	-.42460
30.0	.74840	-3.79614	-1.90075	-1.08000	-.59068	-.28166
35.0	.83985	-3.66117	-1.74443	-.92631	-.46054	-.18896
40.0	.92557	-3.53795	-1.60758	-.79984	.36270	-.12853
45.0	1.00665	-3.42440	-1.48659	-.69462	-.28827	-.08874
50.0	1.08377	-3.31912	-1.37888	-.60637	-.23108	.06222
55.0	1.15737	-3.22112	-1.28252	.53190	.18675	.04431
60.0	1.22776	-3.12968	-1.19606	-.46876	.15213	.03205

Table B.1 (Continued)

$M = 8$

ψ_0	a_1^1	a_2^1	a_3^1	a_4^1	a_5^1	a_6^1	a_7^1	a_8^1
10.0	.27263	-4.54786	-2.87169	-2.20013	-1.78436	-1.47550	-1.22456	-1.01102
15.0	.39001	-4.35664	-2.61304	-1.88265	-1.41855	-1.07327	-.79881	-.57510
20.0	.49035	-4.19615	-2.40201	-1.63354	-1.14575	-.79211	-.52447	-.32158
25.0	.57873	-4.05762	-2.22544	-1.43396	-.93949	-.59504	-.35039	-.18090
30.0	.65896	-3.93439	-2.07326	-1.26942	-.77929	-.45368	-.23830	-.10337
35.0	.73330	-3.82248	-1.93931	-1.13081	-.65210	-.35002	-.16465	-.06024
40.0	.80307	-3.71951	-1.81979	-1.01236	-.54955	-.27272	-.11537	-.03585
45.0	.86906	-3.62400	-1.71222	-.91017	-.46598	-.21435	-.08189	-.02180
50.0	.93179	-3.53494	-1.61485	-.82144	-.39736	-.16986	-.05884	-.01355
55.0	.99160	-3.45161	-1.52638	-.74403	-.34068	-.13567	-.04279	-.00860
60.0	1.04872	-3.37348	-1.44576	-.67626	-.29362	-.10921	-.03150	-.00558

$M = 10$

ψ_0	a_1^1	a_2^1	a_3^1	a_4^1	a_5^1	a_6^1	a_7^1	a_8^1	a_9^1	a_{10}^1
10.0	.26123	-4.56667	-2.89764	-2.23283	-1.82335	-1.52021	-1.27436	-1.06520	-.88273	-.72160
15.0	.36354	-4.39983	-2.67160	-1.95479	-1.50205	-1.16565	-.89735	-.67701	-.49464	-.34450
20.0	.44925	-4.26227	-2.48975	-1.73849	-1.26274	-.91561	-.64889	-.44165	-.28246	-.16348
25.0	.52470	-4.14316	-2.33625	-1.56217	-1.07638	-.73189	-.47925	-.29526	-.16551	-.07889
30.0	.59337	-4.03653	-2.20222	-1.41347	-.92617	-.59214	-.35936	-.20114	-.09925	-.03900
35.0	.65707	-3.93919	-2.08286	-1.28546	-.80249	-.48344	-.27260	-.13908	-.06071	-.01981
40.0	.71685	-3.84928	-1.97526	-1.17386	-.69929	-.39765	-.20880	-.09740	-.03780	-.01035
45.0	.77336	-3.76561	-1.87750	-1.07577	-.61239	-.32927	-.16134	-.06902	-.02393	-.00556
50.0	.82701	-3.68738	-1.78823	-.98905	-.53875	-.27434	-.12571	-.04946	-.01540	-.00370
55.0	.87811	-3.61400	-1.70641	-.91207	-.47603	-.22995	-.09875	-.03585	-.01007	-.00174
60.0	.92686	-3.54503	-1.63125	-.84352	-.42240	-.19389	-.07821	-.02628	-.00669	-.00101

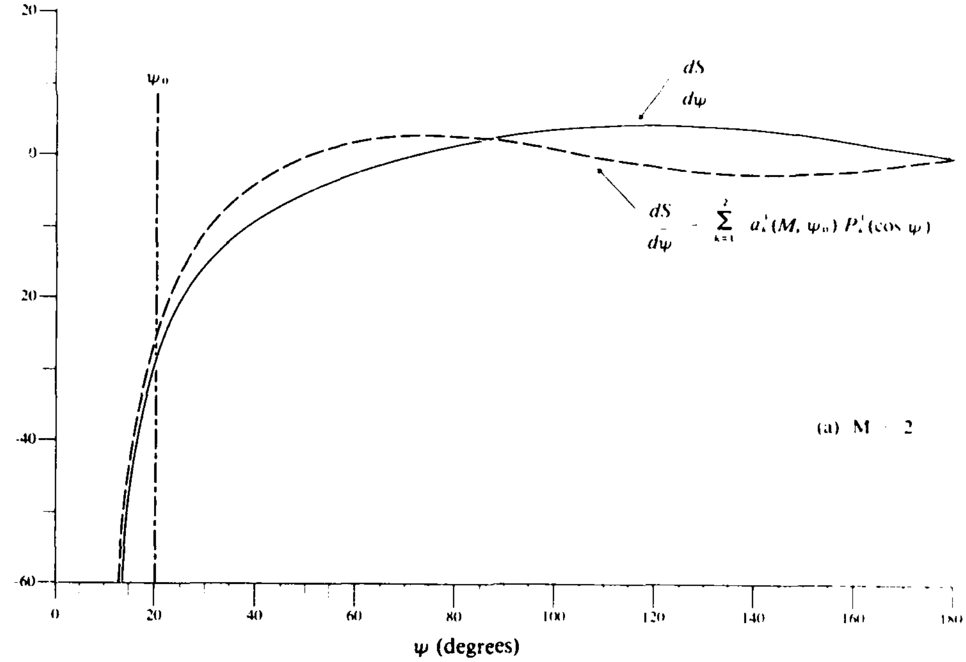


Figure B.1 Vening Meinesz Kernel and Modified Kernel Based on a Molodenskii Type Procedure ($\psi_0 = 20°$)

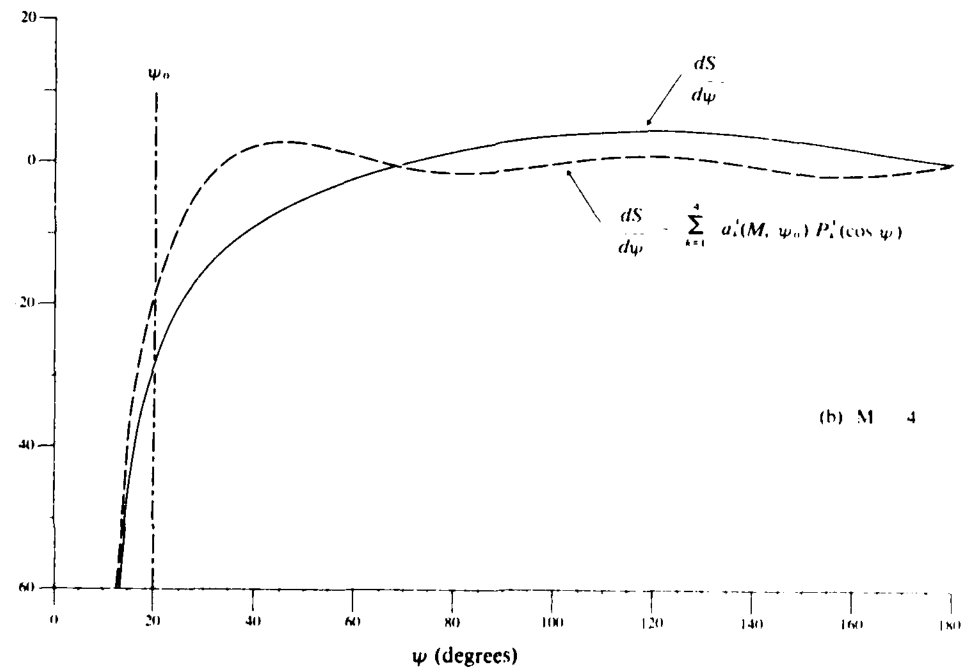


Figure B.1 Vening Meinesz Kernel and Modified Kernel Based on a Molodenskii Type Procedure ($\psi_0 = 20°$)

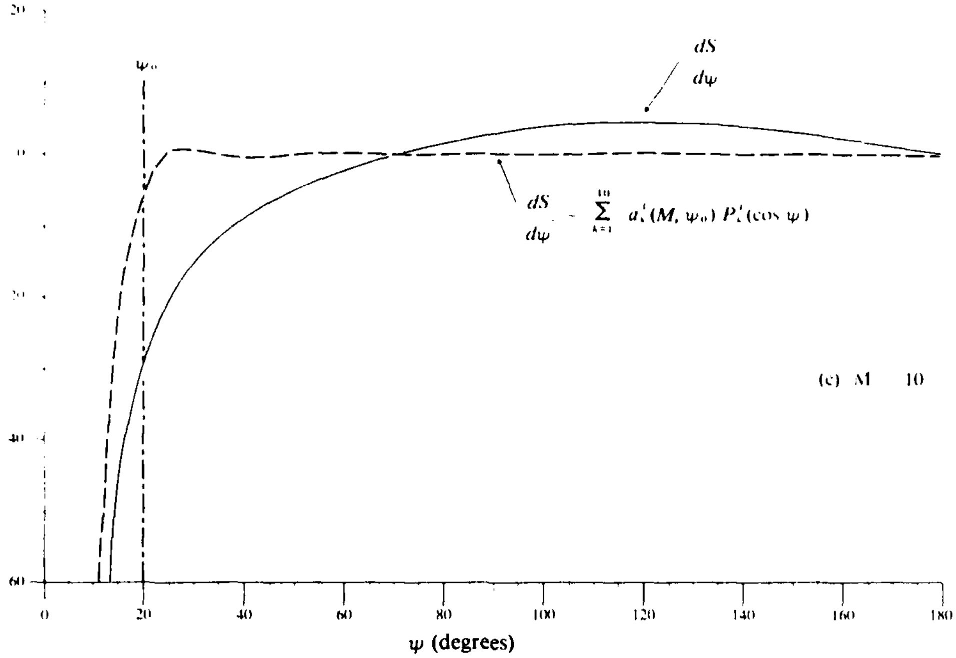


Figure B.1 Vening Meinesz Kernel and Modified Kernel Based on a Molodenskii Type Procedure (ψ_0 = 20°)

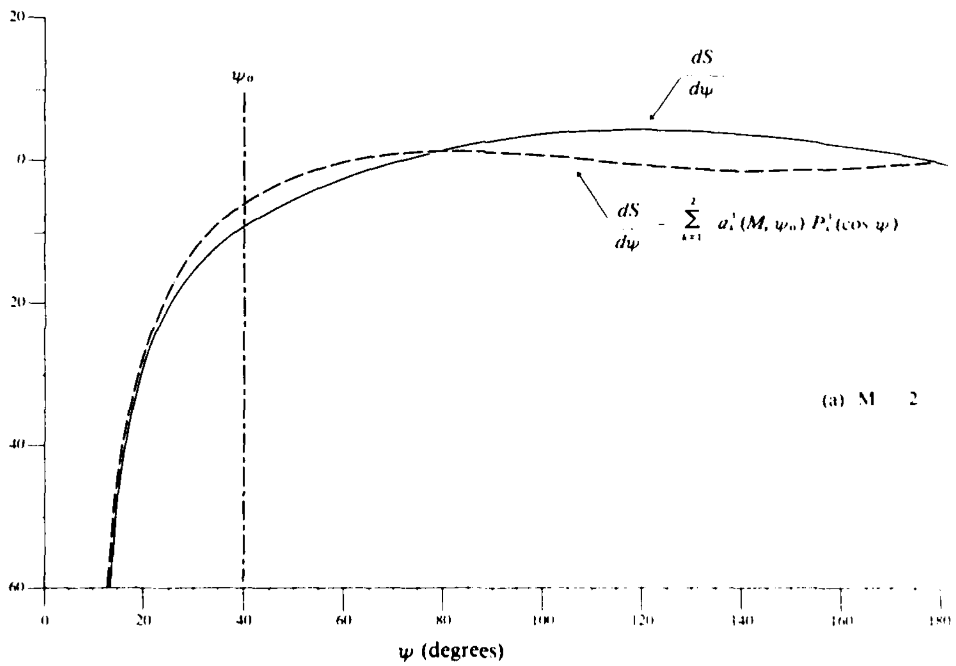


Figure B.2 Vening Meinesz Kernel and Modified Kernel Based on a Molodenskii Type Procedure (ψ_0 = 40°)

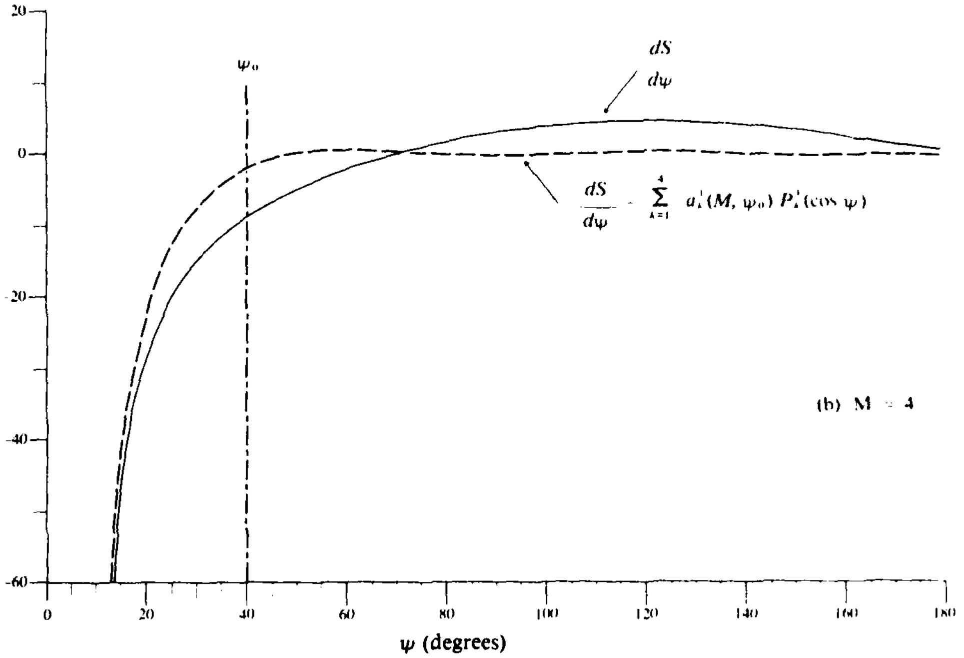


Figure B.2 Vening Meinesz Kernel and Modified Kernel Based on a Molodenskii Type Procedure (ψ_0 = 40°)

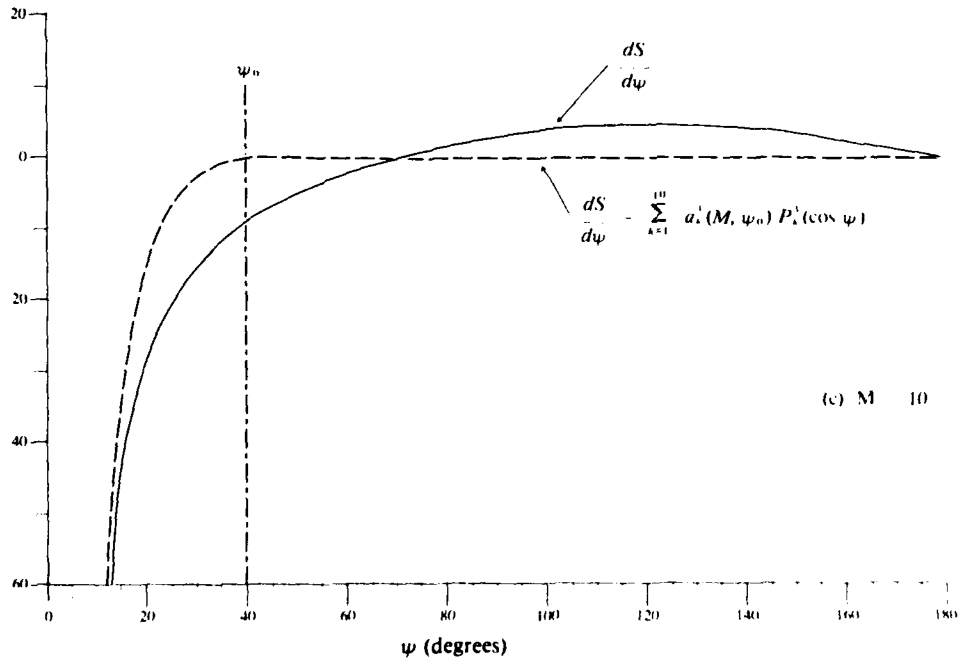
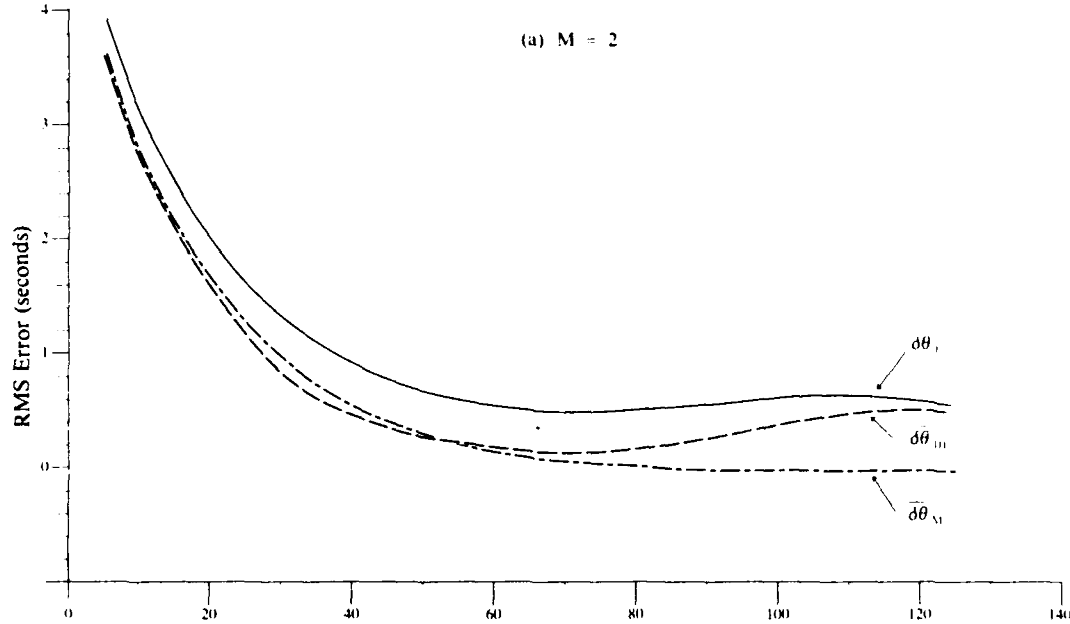
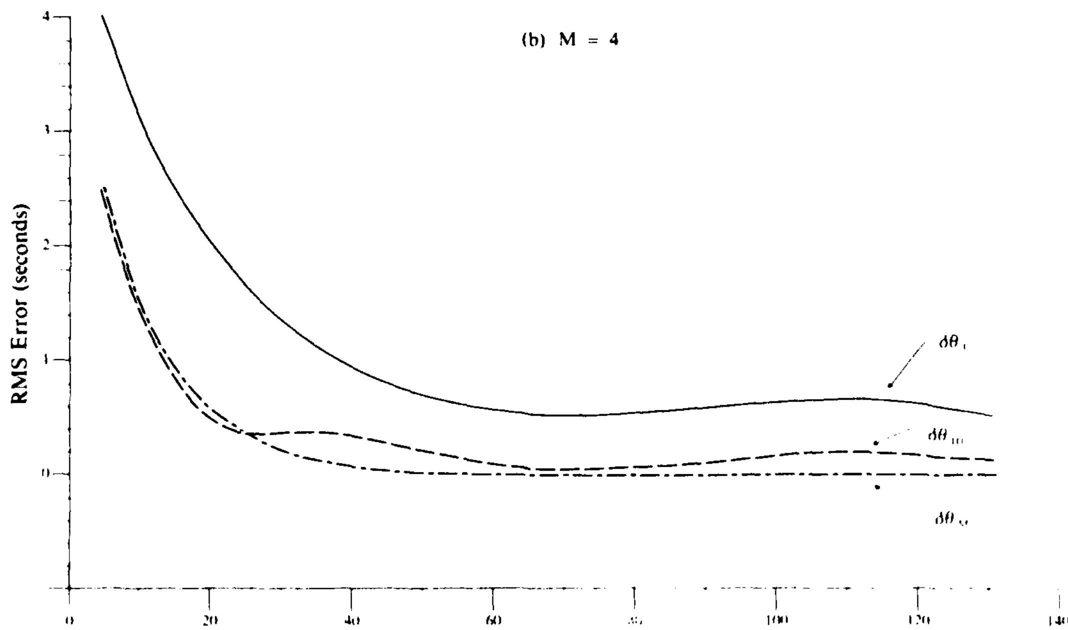


Figure B.2 Vening Meinesz Kernel and Modified Kernel Based on a Molodenskii Type Procedure (ψ_0 = 40°)



CAP SIZE ψ_0 (degrees)

Figure B.3 Expected Truncation Error (RMS) for Vertical Deflection Using Method III and a Molodenskii Type Procedure



CAP SIZE ψ_0 (degrees)

Figure B.3 Expected Truncation Error (RMS) for Vertical Deflection Using Method III and a Molodenskii Type Procedure

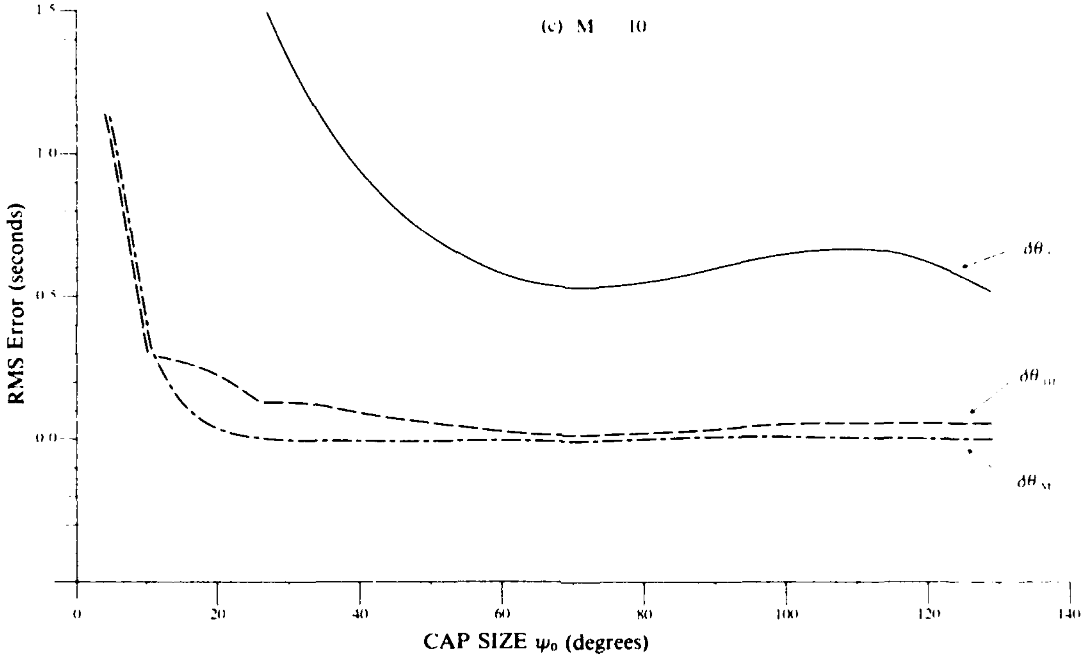


Figure B.3 Expected Truncation Error (RMS) for Vertical Deflection Using Method III and a Molodenskii Type Procedure

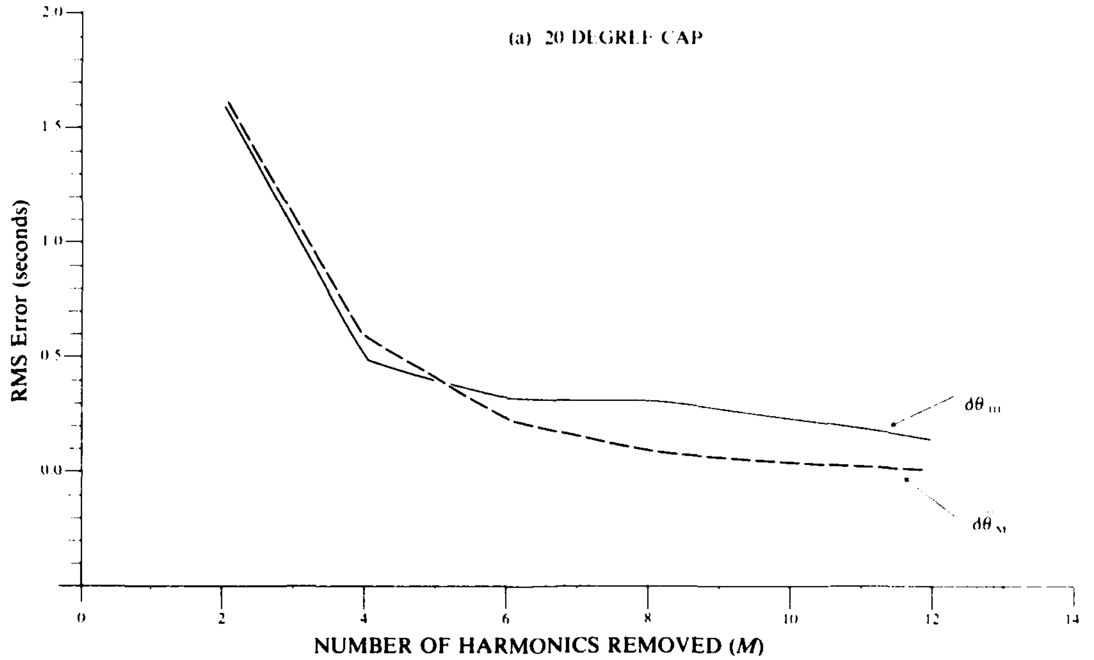


Figure B.4 Expected Truncation Error (RMS) for Vertical Deflection Using Method III and a Molodenskii Type Procedure as a Function of the Number of Harmonics Removed

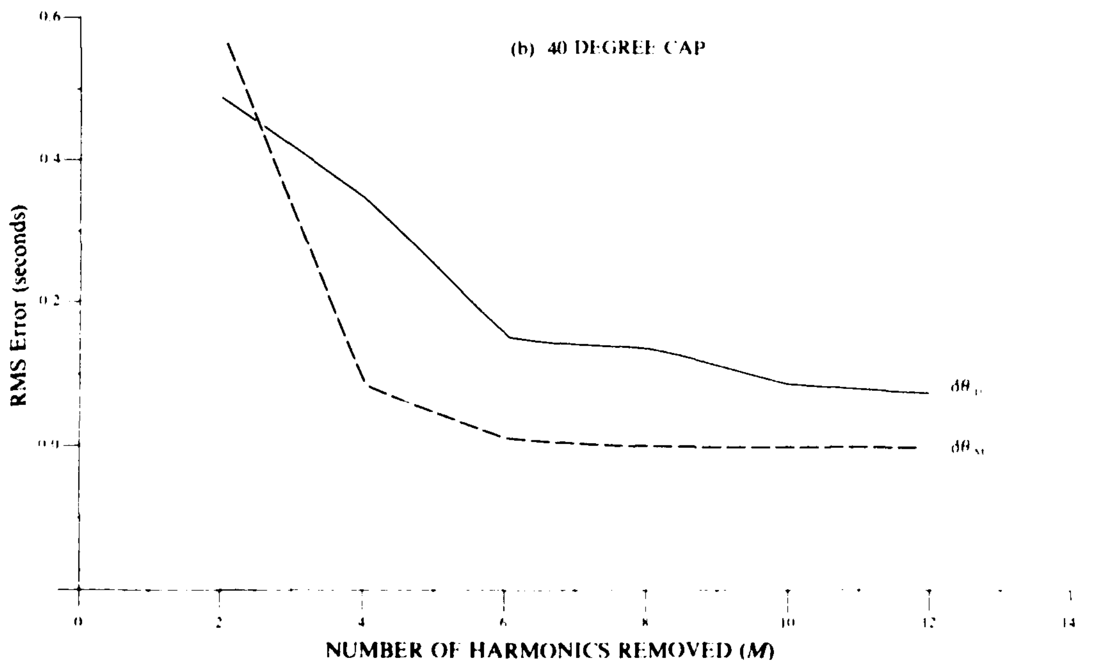


Figure B.4 Expected Truncation Error (RMS) for Vertical Deflection Using Method III and a Molodenskii Type Procedure as a Function of the Number of Harmonics Removed.

TABLE B.2 Expected Truncation Error (seconds) for Vertical Deflection Based on Method III and a Molodenskii Type Procedure

$M = 2$

CAP SIZE ψ_0	$\overline{\delta\theta}_I$	$\overline{\delta\theta}_{III}$	$\overline{\delta\theta}_M$
5.0	3.978880	3.662607	3.663817
10.0	3.111038	2.742907	2.752567
15.0	2.505130	2.096936	2.126250
20.0	2.031197	1.590299	1.648072
25.0	1.652192	1.186509	1.273362
30.0	1.352733	.873341	.978800
35.0	1.121781	.644263	.748134
40.0	.947027	.489385	.568485
45.0	.814783	.390303	.429320
50.0	.712961	.323158	.322058
55.0	.634082	.269139	.239799
60.0	.575874	.221037	.177051
65.0	.539023	.182188	.129476
70.0	.523560	.161844	.093660
75.0	.526517	.166677	.066921
80.0	.542567	.193653	.047152
85.0	.566245	.235818	.032702
90.0	.593267	.288449	.022278
95.0	.620287	.347719	.014873
100.0	.643988	.408531	.009703

$M = 4$

CAP SIZE ψ_0	$\overline{\delta\theta}_I$	$\overline{\delta\theta}_{III}$	$\overline{\delta\theta}_M$
5.0	3.978880	2.490406	2.496183
10.0	3.111038	1.469157	1.508087
15.0	2.505130	.842184	.931024
20.0	2.031197	.479325	.575539
25.0	1.652192	.364583	.355742
30.0	1.352733	.379008	.219870
35.0	1.121781	.383924	.135749
40.0	.947027	.347850	.083569
45.0	.814783	.283567	.051187
50.0	.712961	.211813	.031125
55.0	.634082	.148805	.018749
60.0	.575874	.101453	.011165
65.0	.539023	.069449	.006559
70.0	.523560	.053782	.003792
75.0	.526517	.056723	.002153
80.0	.542567	.070400	.001197
85.0	.566245	.088669	.000650
90.0	.593267	.113041	.000344
95.0	.620287	.143649	.000176
100.0	.643988	.174213	.000087

Table B.2 (Continued)

$M = 6$

CAP SIZE ψ_0	$\overline{\delta\theta}_{I}$	$\overline{\delta\theta}_{III}$	$\overline{\delta\theta}_{VI}$
5.0	3.978880	1.855658	1.869405
10.0	3.111038	.841762	.913194
15.0	2.505130	.377418	.454259
20.0	2.031197	.321540	.227998
25.0	1.652192	.343545	.115407
30.0	1.352733	.299329	.058705
35.0	1.121781	.218555	.029874
40.0	.947027	.154355	.015147
45.0	.814783	.131300	.007625
50.0	.712961	.120768	.003799
55.0	.634082	.099153	.001869
60.0	.575874	.069989	.000905
65.0	.539023	.044329	.000430
70.0	.523560	.030527	.000200
75.0	.526517	.033051	.000091
80.0	.542567	.043041	.000040
85.0	.566245	.052876	.000017
90.0	.593267	.066097	.000007
95.0	.620287	.085426	.000003
100.0	.643988	.102787	.000001

$M = 8$

CAP SIZE ψ_0	$\overline{\delta\theta}_{I}$	$\overline{\delta\theta}_{III}$	$\overline{\delta\theta}_{VI}$
5.0	3.978880	1.419918	1.444878
10.0	3.111038	.484336	.571066
15.0	2.505130	.283755	.231284
20.0	2.031197	.304085	.095652
25.0	1.652192	.238663	.040132
30.0	1.352733	.157320	.016938
35.0	1.121781	.138061	.007142
40.0	.947027	.135345	.002994
45.0	.814783	.107694	.001243
50.0	.712961	.073710	.000509
55.0	.634082	.055312	.000205
60.0	.575874	.043586	.000081
65.0	.539023	.028934	.000031
70.0	.523560	.018118	.000012
75.0	.526517	.020591	.000004
80.0	.542567	.029886	.000002
85.0	.566245	.037026	.000001
90.0	.593267	.045138	.000000
95.0	.620287	.058739	.000000
100.0	.643988	.068533	.000000

Table B.2 (Continued)

$M = 10$

CAP SIZE ψ_0	$\overline{\delta\theta}_I$	$\overline{\delta\theta}_{III}$	$\overline{\delta\theta}_{VI}$
5.0	3.978880	1.094937	1.133355
10.0	3.111038	.303695	.363392
15.0	2.505130	.279978	.121074
20.0	2.031197	.226822	.041608
25.0	1.652192	.140284	.014535
30.0	1.352733	.133846	.005102
35.0	1.121781	.122272	.001786
40.0	.947027	.086117	.000620
45.0	.814783	.071222	.000213
50.0	.712961	.063705	.000072
55.0	.634082	.044521	.000024
60.0	.575874	.028111	.000008
65.0	.539023	.018260	.000002
70.0	.523560	.010516	.000001
75.0	.526517	.012848	.000000
80.0	.542567	.021672	.000000
85.0	.566245	.028007	.000000
90.0	.593267	.033488	.000000
95.0	.620287	.043663	.000000
100.0	.643988	.048752	.000000

$M = 12$

CAP SIZE ψ_0	$\overline{\delta\theta}_I$	$\overline{\delta\theta}_{III}$	$\overline{\delta\theta}_{VI}$
5.0	3.978880	.843845	.896296
10.0	3.111038	.248106	.234207
15.0	2.505130	.247209	.064694
20.0	2.031197	.145935	.018544
25.0	1.652192	.125036	.005402
30.0	1.352733	.113842	.001579
35.0	1.121781	.078850	.000459
40.0	.947027	.074463	.000132
45.0	.814783	.060512	.000037
50.0	.712961	.042577	.000010
55.0	.634082	.035972	.000003
60.0	.575874	.024122	.000001
65.0	.539023	.012894	.000000
70.0	.523560	.006329	.000000
75.0	.526517	.008318	.000000
80.0	.542567	.016099	.000000
85.0	.566245	.022160	.000000
90.0	.593267	.026173	.000000
95.0	.620287	.034066	.000000
100.0	.643988	.036357	.000000

APPENDIX C

DERIVATION OF CERTAIN INTEGRALS INVOLVING LEGENDRE FUNCTIONS

DERIVATION OF CERTAIN INTEGRALS INVOLVING LEGENDRE FUNCTIONS

Use will be made of the recurrence relations as given by Lebedev (1972). In all cases the integration interval will be $-1 \leqslant a \leqslant x \leqslant b \leqslant 1$.

I. Definitions:

$$P_n^m(x) = (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x)$$

$$P_n'(x) = \frac{d}{dx} P_n(x) \tag{C.1}$$

II. Recurrence relations:

$$(1 - x^2) \frac{d}{dx} P_n^m = (n + m) P_{n-1}^m - nx P_n^m \tag{C.2}$$

$$nP_n = xP_n' - P_{n-1}' \tag{C.3}$$

$$nP_{n-1} = P_n' - xP_{n-1}' \tag{C.4}$$

III. Theorems:

(i) Theorem C-1:

$$\int_a^b P_n^m(x)\, P_k^m(x)\, dx = \left[\frac{(n - k)\, xP_n^m P_k^m - (n + m)\, P_{n-1}^m P_k^m + (k + m)\, P_n^m P_{k-1}^m}{(n - k)(n + k + 1)}\right]_a^b \tag{C.5}$$

for $n \neq k$ and $n \geqslant 1$, $k \geqslant 1$.

Proof: Consider the following differential equations satisfied by the associated Legendre functions

$$\frac{d}{dx}\{(1 - x^2)y'\} + \left\{n(n + 1) - \frac{m^2}{1 - x^2}\right\} y = 0 \tag{C.6}$$

with a solution given by $P_n^m(x)$, and

$$\frac{d}{dx}\{(1 - x^2)y'\} + \left\{k(k + 1) - \frac{m^2}{1 - x^2}\right\} y = 0 \tag{C.7}$$

with one of its solutions $P_k^m(x)$. Assume that $n \neq k$ and multiply equation (C.6) by $P_k^m(x)$ and (C.7) by $P_n^m(x)$, form their difference, and then integrate to get

$$\int_a^b P_k^m \left\{\frac{d}{dx}(1 - x^2)\frac{d}{dx} P_n^m\right\} dx - \int_a^b P_n^m \frac{d}{dx}\left\{(1 - x^2)\frac{d}{dx} P_k^m\right\} dx$$

$$+ \{n(n + 1) - k(k + 1)\} \int_a^b P_n^m P_k^m\, dx = 0. \tag{C.8}$$

Integrating the first two integrals by parts and noting that

$n(n+1) - k(k+1) = (n-k)(n+k+1)$

leads to

$$\int_a^b P_n^m P_k^m \, dx = \left[\frac{(1-x^2)}{(n-k)(n+k+1)} \left\{ P_n^m \frac{d}{dx} P_k^m - P_k^m \frac{d}{dx} P_n^m \right\} \right]_a^b \tag{C.9}$$

Using (C.2) and rearranging terms, gives

$$\int_a^b P_n^m P_k^m \, dx = \left[\frac{(n-k)\, x P_n^m P_k^m - (n+m)\, P_{n-1}^m P_k^m + (k+m)\, P_n^m P_{k-1}^m}{(n-k)(n+k+1)} \right]_a^b$$

for $n \neq k$ and $n \geq 1$, $k \geq 1$.

(ii) Theorem C-2:

$$\int_a^b P_n^1 P_k^1 \, dx = -nk \left[\frac{(n-k)\, x P_n P_k - (n+1)\, P_n P_{k-1} + (k+1)\, P_{n-1} P_k}{(n-k)(n+k+1)} \right]_a^b$$

for $n \neq k$ and $n \geq 2$, $k \geq 2$. (C.10)

Proof: Set $m = 1$ in equation (C.5) to get

$$\int_a^b P_n^1 P_k^1 \, dx = \left[\frac{(n-k)\, x P_n^1 P_k^1 - (n+1)\, P_{n-1}^1 P_k^1 + (k+1)\, P_n^1 P_{k-1}^1}{(n-k)(n+k+1)} \right]_a^b . \tag{C.11}$$

From equation (C.1) for $m = 1$, and (C.3)

$$P_j^1 = (1-x^2)^{1/2} P_j'$$

$$P_{j-1}^1 = (1-x^2)^{1/2} P_{j-1}'$$

$$= (1-x^2)^{1/2} (x P_j' - j P_j), \text{ for } j \geq 2.$$

Substituting the above into (C.11) and simplifying yields

$$\int_a^b P_n^1 P_k^1 \, dx = \left[(1-x^2) \left(\frac{n(n+1)\, P_n P_k' - k(k+1)\, P_n' P_k}{(n-k)(n+k+1)} \right) \right]_a^b . \tag{C.12}$$

Finally, using equation (C.2) with $m = 0$, gives

$$\int_a^b P_n' P_k' \, dx = -nk \left[\frac{(n-k)\, xP_nP_k - (n+1)\, P_nP_{k-1} + (k+1)\, P_{n-1}P_k}{(n-k)(n+k+1)} \right]_a^b$$

for $n \neq k$ and $n \geqslant 2$, $k \geqslant 2$.

(iii) Theorem C-3:

$$\int_a^b P_n(x)\, P_k(x)\, dx = \left[\frac{(n-k)\, xP_nP_k - nP_{n-1}P_k + k\, P_nP_{k-1}}{(n-k)(n+k+1)} \right]_a^b \tag{C.13}$$

for $n \neq k$, $n \geqslant 1$, $k \geqslant 1$.

Proof: Set $m = 0$ in equation (C.5) and the result follows.

(iv) Theorem C-4:

$$\int_a^b P_n^2 \, dx = \frac{(2n-1)}{(2n+1)} \int_a^b P_{n-1}^2 \, dx + \left[\frac{x(P_n^2 + P_{n-1}^2) - 2P_nP_{n-1}}{(2n+1)} \right]_a^b \tag{C.14}$$

Proof: Replacing one of the $P_n(x)$'s by (C.3) gives

$$n \int_a^b P_n^2 \, dx = \int_a^b xP_nP_n' \, dx - \int_a^b P_nP_{n-1}' \, dx. \tag{C.15}$$

Integrating by parts the integrals on the right side of (C.15) give

$$\int_a^b xP_nP_n' \, dx = \frac{1}{2} \left[xP_n^2\right]_a^b - \frac{1}{2} \int_a^b P_n^2 \, dx \tag{C.16}$$

and

$$\int_a^b P_nP_{n-1}' \, dx = \left[P_nP_{n-1}\right]_a^b - \int_a^b P_{n-1}P_n' \, dx$$

or using identity (C.4)

$$\int_a^b P_nP_{n-1}' \, dx = \left[P_nP_{n-1}\right]_a^b - n \int_a^b P_{n-1}^2 \, dx - \int_a^b xP_{n-1}P_{n-1}' \, dx. \tag{C.17}$$

Substituting (C.16) and (C.17) into (C.15) and simplifying gives

$$(2n+1) \int_a^b P_n^2 \, dx = \left[xP_n^2 - 2P_nP_{n-1}\right]_a^b + 2n \int_a^b P_{n-1}^2 \, dx + 2 \int_a^b xP_{n-1}P_{n-1}' \, dx. \tag{C.18}$$

Integrating the last integral of (C.18) by parts gives

$$\int_a^b xP_{n-1}P_{n-1}' \, dx = \frac{1}{2}\left[xP_{n-1}^2\right]_a^b - \frac{1}{2} \int_a^b P_{n-1}^2 \, dx \tag{C.19}$$

so equation (C.18) can be written as

$$(2n+1)\int_a^b P_n^2\,dx = (2n-1)\int_a^b P_{n-1}^2\,dx + [x(P_n^2 + P_{n-1}^2) - 2P_nP_{n-1}]_a^b \tag{C.20}$$

or

$$\int_a^b P_n^2\,dx = \frac{(2n-1)}{(2n+1)}\int_a^b P_{n-1}^2\,dx + \left[\frac{x(P_n^2 + P_{n-1}^2) - 2P_nP_{n-1}}{(2n+1)}\right]_a^b .$$

(v) Theorem C.5:

$$\int_a^b [P_n^1]^2\,dx = n(n+1)\int_a^b P_n^2\,dx + n[P_nP_{n-1} - xP_n^2]_a^b \quad \text{for } n \geqslant 2. \tag{C.21}$$

Proof: From (C.1) with $m = 1$ and (C.2) with $m = 0$

$$[P_n^1]^2 = P_n'(1-x^2)\,P_n'$$

$$= P_n^1(nP_{n-1} - nxP_n).$$

So

$$\int_a^b [P_n^1]^2\,dx = n\int_a^b P_{n-1}P_n'\,dx - n\int_a^b xP_nP_n'\,dx. \tag{C.22}$$

Using (C.4) and then (C.19) in the first integral on the right of (C.22) yields

$$\int_a^b P_{n-1}P_n'\,dx = n\int_a^b P_{n-1}^2\,dx + \int_a^b xP_{n-1}P_{n-1}'\,dx$$

$$= (n - \tfrac{1}{2})\int_a^b P_{n-1}^2\,dx + \tfrac{1}{2}\,[xP_{n-1}^2]_a^b. \tag{C.23}$$

Using equation (C.14) in equation (C.23) gives

$$\int_a^b P_{n-1}P_n'\,dx = (n + \tfrac{1}{2})\int_a^b P_n^2\,dx + [P_nP_{n-1} - \tfrac{1}{2}\,xP_n^2]_a^b . \tag{C.24}$$

Finally, substituting equations (C.16) and (C.24) into (C.22) and simplifying yields

$$\int_a^b [P_n^1]^2\,dx = n(n+1)\int_a^b P_n^2\,dx + n[P_nP_{n-1} - xP_n^2]_a^b .$$

DISTRIBUTION

Strategic Systems Project Office (SP-23)
Washington, DC 20376

Defense Technical Information Center
Cameron Station
Alexandria, VA 22314 (12)

Library of Congress
Washington, DC 20540
Attn: Gift and Exchange Division (4)

Director Defense Mapping Agency
U.S. Naval Observatory
Bldg. 56
Washington, DC 20360
Attn: O. W. Williams
C. Martin
P. M. Schwimmer
D. A. Rekenthaler

Commander
Naval Oceanographic Office
NSTL Station
Bay St. Louis, MS 39522
Attn: T. Davis
J. Hankins

Office of the Oceanographer
Hoffman 2
200 Stovall Street
Alexandria, VA 22332
Attn: CAPT D. Brown

Oceanographer of the Navy
Hoffman 2
200 Stovall St.
Alexandria, VA 22332

Defense Mapping Agency
Aerospace Center
St. Louis, MO 63118
Attn: R. Ballew
J. Finkland (Code GDG) (2)
M. Schultz (Code GDGA) (4)

Defense Mapping Agency
Hydrographic/Topographic Center
Washington, DC 20315
Attn: R. Smith (3)

Air Force Geophysical Laboratory
L. G. Hanscom Field
Bedford, MA 01730
Attn: G. Hadgigeorge

U.S. Army Topographic Laboratory
Ft. Belvoir, VA 22060

NOAA/National Ocean Survey/National Geodetic Survey
6001 Executive Blvd.
Rockville, MD 20852
Attn: B. H. Chovitz
T. Soler

Battelle Columbus Laboratories
505 King Avenue
Columbus, OH 43201
Attn: A. G. Mourad
S. Gopalapillai

Smithsonian Astrophysical Observatory
60 Garden Street
Cambridge, MA 02138
Attn: E. M. Gaposchkin

U.S. Naval Postgraduate School
Monterey, CA 93940
Attn: Library

U.S. Naval Academy
Annapolis, MD 21402
Attn: Technical Library

The Ohio State University
Department of Geodetic Science
164 W. Nineteenth Avenue
Columbus, OH 43210
Attn: Dr. I. I. Mueller
Dr. U. A. Uotila
Dr. R. Rapp

Aerospace Corporation
2350 East El Segundo Boulevard
El Segundo, CA
Attn: R. Farrar
L. Wong

Department of Civil Engineering
Purdue University
Lafayette, IN
Attn: Prof. L. Kivioja

Technishe Hochschule Graz
Technikerstr. 4, A-8010 Graz
Austria
Attn: Prof. P. Meissl
Prof. H. Moritz
Prof. K. Rinner

Geodetic Survey of Canada
615 Booth St., Ottawa, Canada
Attn: J. Kouba

Department of Survey Engineering
University of New Brunswick
Fredericton, N.B., Canada
Attn: Dr. P. Vanicek

Geodetic Institute of the Delft
Technological University
Delft, Netherlands
Attn: Dr. B. Van Gelder

Local:
E41
K10 (50)
K50 (10)
K05

DATE
ILMED
-8